A COUNTEREXAMPLE CONCERNING INSEPARABLE FIELD EXTENSIONS

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Abstract. Let $K \supseteq M \supseteq k$ be a chain of fields of characteristic $p \neq 0$ where $K$ is separable over $M$ and $M$ is purely inseparable over $k$. Recently it has been shown that if $K$ has a separating transcendence basis over $M$ or if $M$ is of bounded exponent over $k$, then $K = M \otimes_k S$ where $S$ is separable over $k$. This note presents an example to show that, in general, no such $S$ need exist.

Throughout, we consider a chain of fields $K \supseteq M \supseteq k$ of characteristic $p \neq 0$ where $K$ is separable over $M$ and $M$ is purely inseparable over $k$. Recent papers [1] and [2], have examined the question of when $K$ can be expressed as $M \otimes_k S$ where $S$ is a separable extension of $k$. It has been shown that if $M$ is of bounded exponent over $k$ [1, Theorem 5], or if $K$ has a separating transcendence basis over $M$ [2, Lemma 4], then $K = M \otimes_k S$ for some $S$. The purpose of this note is to provide an example to show that, in general, no such $S$ exists. Necessarily, $M$ will be of unbounded exponent over $k$ and $K$ will not have a separating transcendence basis over $M$.

**Example 1.** Let $P$ be a perfect field of characteristic $p \neq 0$ and let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be an algebraically independent set over $P$. Set $K = P(x_1, x_2, \ldots, x_n, \ldots)$, $M = P(x_1x_1^{p^2}, x_2x_1^{p^2}, \ldots, x_nx_1^{p^2}, \ldots)$, $k = P(x_1x_1^{p^2}, x_2x_1^{p^2}, \ldots, x_nx_1^{p^2}, \ldots)$. Since $(x_1x_1^{p^2}, x_2x_1^{p^2}, \ldots, x_nx_1^{p^2}, \ldots)$ is a $p$-basis for $M$ and remains $p$-independent in $K$, so $K$ is separable over $M$. Moreover, elementary calculations show $\{x_1x_1^{p^2}, x_2x_1^{p^2}, \ldots, x_nx_1^{p^2}, \ldots\}$ is actually a $p$-basis for $K$, and thus $K$ is relatively perfect over $M$, i.e. $K = M(K^p)$. We now assume there exists a field $S$ separable over $k$ such that $K = M \otimes_k S$.

**Lemma 2.** $S$ is relatively perfect over $k$.

Proof. Recall that $K = M(K^p)$. Since we are assuming $K = M(S)$, $K^p = M^p(S^p)$, and so $K = M(M^p(S^p)) = M(S^p)$. Thus $K = M \otimes_k k(S^p)$ and we must have $S = k(S^p)$.

Now since $S$ is relatively perfect over $k$, $S = k(S^p) \subseteq k(K^p)$ for all $n$. Thus $S \subseteq \cap k(K^p)$.

**Lemma 3.** $\cap k(K^p) \subseteq P(x_1^{p^2}, x_2^{p^2}, \ldots, x_n^{p^2}, \ldots) = \bar{k}$.

Proof. Since $\bar{k} \supseteq k$, $\cap k(K^p) \subseteq \cap \bar{k}(K^{p^2})$. Since...
\[ K = \overline{k(x_1)} \otimes_{\overline{k}} \overline{k(x_2)} \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} \overline{k(x)} \otimes_{\overline{k}} \cdots, \]
\[ \cap \overline{k(K^p)} = \overline{k}, \text{ and the lemma is established.} \]

We now have \( S \subseteq \cap k(K^p) \subseteq \overline{k}. \) To show no such \( S \) exists it suffices to show \( M(\overline{k}) \neq K. \)

\[ M(\overline{k}) = P(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \left( x_1^p, x_2^p, \ldots, x_n^p, \ldots \right) \]
\[ = P(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \left( x_1^p \right) = M(x_1^p). \]

\( P(x_1, \ldots, x_n) \) is algebraic over \( P(x_1, x_1^p, \ldots, x_{n-1}^p) \), and hence both fields have the same transcendence degree \( n \) over \( P \), which means that \( x_1, x_1x_2^p, \ldots, x_{n-1}x_n^p \) are algebraically independent over \( P \). Since this is true for all \( n \), the set \( \{ x_1, x_1x_2^p, x_2x_3^p, \ldots \} \) is algebraically independent over \( P \), and hence \( x_1 \) is transcendental over \( M(\overline{k}) \subseteq M(x_1) \subseteq K. \) Thus no such \( S \) can exist.

**References**


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