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THE GENERAL SOLUTION OF A FIRST ORDER DIFFERENTIAL POLYNOMIAL

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ABSTRACT. A purely algebraic proof is given of a theorem, proved analytically by Ritt, which determines the number of derivations needed to find a basis for the perfect ideal of the general solution of an algebraically irreducible first order differential polynomial.

1. It was shown by J. F. Ritt [2, 129] that to separate the singular components from the general solution of an irreducible first order differential polynomial G of degree m in the first derivative of the indeterminate it is sufficient to decompose the system formed by G and its first $m-1$ derivatives treated as algebraic polynomials. His proof uses analytic methods. Kolchin [1, p. xiv] observed that this result is valid in abstract differential algebra because of the "differential Lefschetz principle" due to Seidenberg [3, p. 160, embedding theorem], but that no purely algebraic proof is known. In this note I give such a proof of Ritt's theorem. The mechanics of the argument are the same as in Ritt's work, except that I have preferred to introduce a parameter to permit power series with integral rather than fractional exponents.

2. Let K be an ordinary differential field of characteristic 0 with derivation D, and let Y be a differential indeterminate over K. Derivatives in $K\{Y\}$ will frequently be denoted by subscripts.

THEOREM. *Let $G \in K\{Y\}$ be of order* 1 *and degree m in Y_1. Let G have no factor of order* 0 *and no factor in common with $\partial G/\partial Y_1$. Let $P_1, \ldots, P_k$ be those minimal prime divisors of the ideal $(G, \ldots, G_{m-1})$ of $K[Y, \ldots, Y_m]$ which do not contain $\partial G/\partial Y_1$. Then no solution in a component of order* 0 *of the manifold of G as a differential polynomial annuls any P_i.*

REMARK. If G is irreducible, then $k = 1$ and P_1 is the intersection with $K[Y, \ldots, Y_m]$ of the differential ideal of the general solution of G [2, p. 130]. The components of order 0 are, of course, precisely the singular components. It follows readily that in the general case k is the number of irreducible factors of G, and each P_i is the intersection with $K[Y, \ldots, Y_m]$ of the differential ideal of the general solution of an irreducible factor. It follows

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further that no P_i contains a polynomial of order 0.

PROOF. Let a be a solution in a zero order component of the manifold of G. We shall assume that a annuls some P_i, say P_1, and obtain a contradiction. Note that without loss of generality we may assume $a = 0$. Indeed, if we enlarge K to a differential field K_1 the hypotheses concerning factorization of G remain valid in $K_1\{Y\}$. From standard results concerning the effect of ground field extensions we see that P_1 splits into prime components in $K_1[Y, \ldots, Y_m]$ no one of which contains $\partial G/\partial Y_1$. The analogous theorem of differential algebral [1, Chapter III, Proposition 3] shows that a is in a zero order component of G as a polynomial in $K_1\{Y\}$. Choosing K_1 to contain a and then making the substitution $Y \to Y + a$ accomplishes the desired reduction. Henceforth, let $a = 0$.

We introduce a parameter t and form the power series ring $K[[t]]$ and its quotient field $K((t))$. These are not differential. However, for $f \in K((t))$ define ∂f to be the result of applying the derivation D of K to every coefficient of f, and df/dt to be the formal derivative of f with respect to t. Let $h \in K((t))$. Define $D_h f = \partial f + h\, df/dt$. Then D_h is a derivation of $K((t))$.

Since P_1 contains no polynomial of order 0, $Y\partial G/\partial Y_1 \notin P_1$. By hypothesis P_1 admits the solution $Y_i = 0$, $0 \leqslant i \leqslant m$. It follows that P_1 has a solution $Y_i = f_i$, $0 \leqslant i \leqslant m$, not annulling $Y\partial G/\partial Y_1$, where the f_i are in $K((t))$ and begin with terms of positive degree. Concerning this solution we make the following observations.

(a) The f_i annul $G_1, \ldots, G_{m-1}$. This requirement uniquely determines $f_2, \ldots, f_m$ when f_0 and f_1 are given. This is so since for $1 \leqslant i \leqslant m - 1$, $G_i = Y_{i+1}\partial G/\partial Y_1 +$ terms free of Y_j, $j > i$, and since f_0, f_1 do not annul $\partial G/\partial Y_1$.

(b) Choose h so that $D_h f_0 = f_1$. (This is possible since $df_0/dt \neq 0$.) Regarding $K((t))$ as a differential field with the derivation D_h we see that f_0 is a solution of G as a differential polynomial. Hence, $Y_i = D_h^i f_0$, $0 \leqslant i \leqslant m$, is a solution of the G_i, $0 \leqslant i < m$.

Combining (a) and (b) we have

$$\text{(A)} \qquad f_i = D_h^i f_0, \qquad 0 \leqslant i \leqslant m.$$

Let d denote the degree of the initial term of h. We shall show that $d \leqslant 0$. Let I be the prime differential ideal of $K\{y\}$ with generic zero f_0. (The derivation is, of course, D_h.) Since f_0 actually involves t, I is not of order 0. Since $G \in I$, this shows that I is of order 1 and is the ideal of a component of the manifold of G. If $d > 0$, then it is easy to see that each $D_h^i f_0$, $i = 0, 1, 2, \ldots,$ begins with a term of positive degree in t. This implies that 0 is a solution of I, contradicting the assumption that 0 lies in a component of order 0 of the manifold of G.

REMARK. One could also show $d \leqslant 0$ (that is, that f_1 begins with a term of lower degree than f_0) from the fact that G must satisfy the low power condition with respect to Y. Conversely, noting that the result $d \leqslant 0$ applies to an arbitrary solution of G (as a polynomial in Y and Y_1) in series of positive powers of t, and examining the Newton polygon of G, one can prove the necessity of the low power condition for differential polynomials of first order.

From $d \leqslant 0$ it follows easily that if $f \in K((t))$ has an initial term of degree $a > 0$, then $D_h f$ has an initial term of degree $a + d - 1 < a$. Let f_0, f_1, f_m begin with terms of degrees a, b, and c respectively. Using (A) and the preceding observation we find $a - b = 1 - d$, $a - c = m(1 - d)$. That is, $c = a - m(a - b)$.

G must contain two terms with distinct power products $Y^p Y_1^q$ and $Y^r Y_1^s$ which yield terms of the same degree after the substitutions $Y = t^a$, $Y_1 = t^b$. Otherwise f_0, f_1 could not annul G. Therefore, $pa + qb = ra + sb$. Let, say, $q > s$. ($q = s$ is not possible, since it implies $r = p$.) Putting $q - s = n$ we rewrite the preceding equation as $nb = (r - p)a$. Then $r - p < n$, since $b < a$. Hence, $nb \leqslant (n - 1)a$. By hypothesis $q \leqslant m$, and so $n \leqslant m$. Then $mb \leqslant (m - 1)a$. Together with the result of the preceding paragraph this implies $c \leqslant 0$. This is a contradiction, and the proof is complete.

COROLLARY. *Let G be an algebraically irreducible differential polynomial of $K\{Y\}$ which is of order 1 and degree m in Y_1. Let I be the differential ideal of the general component of the manifold of G. Let P be the unique minimal prime divisor of $(G, \ldots, G_{m-1})$ which does not contain $\partial G/\partial Y_1$. Then $I = \{P\}$, and I has a basis as a perfect differential ideal consisting of polynomials of order not exceeding m.*

PROOF. By [2, p. 30], $I \supset \{P\}$; and, of course, $\{P\} \supset \{G\}$. It follows from the Theorem that no minimal prime divisor of $\{G\}$ except I contains P. Hence $I = \{P\}$. A basis for P as a polynomial ideal is a basis for $\{P\}$ as a perfect differential ideal.

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