SHORTER NOTES

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A SHORT PROOF OF THE UNIQUENESS OF HAAR MEASURE

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The purpose of this note is to give a brief proof of the uniqueness (up to a positive multiple) of left Haar measure \( \mu \) (the existence of which we assume) on an arbitrary Hausdorff locally compact group \( G \). The approach used here, employing the well-known device of an approximate identity, appears to be more transparent than any that we have found in the literature (e.g., [4, Theorem 29D, pp. 115–116] or [1, Theorem 1(B), pp. 15–16]). We remark that the elegant uniqueness proof for the Abelian case [5, 1.1.3, p. 2] cannot be improved upon; thus, our proof is of interest only for non-Abelian locally compact groups. We also observe that it is possible to give a combined existence and uniqueness proof (e.g., [2]).

We begin with some notation. Let \( \nu \) denote a measure on \( G \) and let \( f, g \) be continuous functions on \( G \) with compact support. For such an \( f \), let \( f'(y) = f(y^{-1}) \); also, for \( x \) in \( G \), let \( (f)(y) = f(x^{-1}y) \). The convolution \( f * g \) is defined as usual (using the Haar measure \( \mu \)):

\[
(f * g)(x) = \int_G f(y) g(y^{-1}x) \, d\mu(y) = \mu(f \cdot x'(g'))
\]

and the convolution \( \nu * f \) is the continuous function on \( G \) defined by

\[
(\nu * f)(x) = \int_G f(y^{-1}x) \, d\nu(y) = \nu(x'(f')).
\]

Next, we recall two standard facts. First, there exists a (right) approximate identity \( (g_a) \) consisting of continuous functions on \( G \) with compact support; that is, \( (g_a) \) is a net with the property

\[
\nu(f) = \lim_a \nu(f * g_a),
\]

for every \( f \) and every \( \nu \). Second, by an application of Fubini's Theorem [3, Lemma A.2(iii), p. 179], it follows that

\[
\nu(f * g) = \mu(f \cdot (\nu * g')),
\]

for every \( f, g \) and every \( \nu \).

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Theorem. If \( \nu \) is a left translation invariant measure on \( G \), then \( \nu \) is a complex multiple of \( \mu \).

Proof. Let \( (g_a) \) be an approximate identity; then for every \( f \), (3) obtains. Rewriting (3), using (4), yields

\[
\nu(f) = \lim_{a} \mu(f \cdot (\nu * g_a')),
\]

for every \( f \). However, since \( \nu \) is left translation invariant, we have

\[
(\nu * g_a')(x) = \nu'(g_a'(x)) = \nu(g_a),
\]

for each \( x \) in \( G \) and each \( a \). Consequently, (5) becomes

\[
\nu(f) = \lim_{a} \mu(f \cdot \nu(g_a)) = \lim_{a} \nu(g_a) \cdot \mu(f) = \left( \lim_{a} \nu(g_a) \right) \cdot \mu(f),
\]

for every \( f \). Finally, by choosing \( f \) so that \( \mu(f) \) is nonzero, it follows that \( \lim_{a} \nu(g_a) \) is equal to a constant \( c \) and that \( \nu = c \cdot \mu \). Q.E.D.

Added in proof. We should observe that the existence of a net \( (g_a) \) satisfying (3) does not depend upon the essential uniqueness of Haar measure; in particular, the modular function is not involved. Indeed, if for every compact symmetric neighborhood \( a \) of the identity we let \( g_a \) be a symmetric \((g_a = g_a')\) positive continuous function on \( G \) supported in \( a \) with \( \mu(g_a) = 1 \), then a straightforward argument using the uniform continuity of \( f \) and the left translation invariance of \( \mu \) yields (3).

References


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