ON THE RATE OF GROWTH OF THE WALSH ANTIDIFFERENTIATION OPERATOR

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Abstract. In [1] Butzer and Wagner introduced a concept of differentiation and antiderivative of Walsh-Fourier series. Antidifferentiation is accomplished by convolving (in the sense of the Walsh group) against a function $\Omega$. In this paper we study growth and the continuity properties of $\Omega$ showing that $\Omega$ is bounded from below by $-1$, is continuous in $(0, 1)$ and grows at most like $\log 1/x$ as $x \to 0$. We use this information to study continuity properties of differentiable functions.

Introduction. In [1], Butzer and Wagner introduced a concept of derivative and antiderivative of Walsh-Fourier series. In this paper we are interested primarily in the antiderivative.

Let $W_n$ denote the $n$th Walsh function. Let $\Omega$ be the a.e. defined function whose Walsh-Fourier series is

$$\Omega(x) = 1 + \sum_{K=1}^{\infty} \frac{W_K(x)}{K}$$

($\Omega$ exists as an $L^2$ function since $1/K$ as in $l^2$). Then convolution against $\Omega$ with respect to the Walsh addition on $[0, 1]$ defines an integral operator which in the Butzer-Wagner theory plays the role of antiderivativation.

It is the purpose of this paper to investigate the continuity properties and growth properties of $\Omega$. Our main results are that $\Omega$ is continuous everywhere in $[0, 1)$ except at 0 and at zero it grows at most like $\log 1/x$. Furthermore, we show that $\Omega(x) \geq -1$ for all $x$. This is interesting for, as commented in [2], $\Omega$ is not positive. Hence, the antiderivative of a positive function need not be positive. However, from the above convolution with $1 + \Omega$ is positive and still yields a concept of antiderivative. Hence, it is possible to get a positive antidifferentiation operator.

Our main technique is to compare $\Omega$ with the function

$$\tilde{\Omega}(x) = \sum_{K=0}^{\infty} \frac{W_K}{K+1}(x).$$

The relationship between $\tilde{\Omega}$ and $\Omega$ is simple:

$$|\Omega(x) - \tilde{\Omega}(x)| = \left| \sum_{K=1}^{\infty} \left( \frac{1}{K} - \frac{1}{K+1} \right) W_K(x) \right| \leq \sum_{K=1}^{\infty} \frac{1}{K} - \frac{1}{K+1} = 1.$$
The series telescopes! Furthermore, since the above sum converges absolutely, \( \Omega - \tilde{\Omega} \) is continuous in the Walsh sense so \( \Omega \) will be continuous whenever \( \tilde{\Omega} \) is.

The advantage to studying \( \tilde{\Omega} \) over \( \Omega \) is that we can write down a formula for \( \tilde{\Omega} \). Specifically, we reason as follows. Let \( t \in \mathbb{R}, |t| < 1 \). Let \( P(t, x) = \sum_{K=0}^{\infty} t^K W_K(x) \). This converges absolutely and uniformly in \( |t| \leq \delta < 1 \) for all \( x \).

In this paper we derive the following formula:

\[
P(t, x) = \frac{1}{1 - t} \prod_{K=0}^{\infty} \left( 1 - \frac{t^{2^K}}{1 + t^{2^K}} \right)
\]

where \( x \in (0, 1) \) and \( x = \sum_{K=0}^{\infty} 2^{-i_K-1} \) is the diadic expansion of \( x \) (if \( x \) is a diadic rational we take the finite expansion). From this formula, \( \lim_{t \to 1^-} P(t, x) \) exists. Hence, \( \lim_{t \to 1^-} \int_0^1 P(s, x) \, ds \) exists. It follows that \( \sum W_K(x)/(K+1) \) is Abel summable. Furthermore, \( K W_K(x)/(K+1) \) converges uniformly for all \( K \). It follows from the Hardy-Littlewood theorem [3, Theorem 4.22] that \( \sum W_K(x)/(K+1) \) converges for all \( x \in (0, 1) \), and equals \( \int_0^1 P(t, x) \). This allows us to obtain our explicit estimates on \( \tilde{\Omega} \).

We begin with the proof of (1) above. We consider the \( W_n \) as having been extended periodically to all of \( \mathbb{R} \). Let

\[
P_N(t, x) = \sum_{K=0}^{2^N-1} t^K W_K(x).
\]

**LEMMA 1.** If \( N \geq i \geq 0 \),

\[
P_N(t, x + 2^{-(i+1)}) = P_i(t, x) P_{N-i}(-t^{2^i}, 2^i x).
\]

**PROOF.**

\[
\begin{align*}
P_N(t, x + 2^{-(i+1)}) &= \sum_{K=0}^{2^N-1} t^K W_K(2^{-(i+1)}) W_K(x) \\
&= \sum_{i=0}^{2^N-i-1} \sum_{m=0}^{2^i-1} (W_{m+2^i l}(2^{-(i+1)}) W_{m+2^i l}(x) t^{m+2^i}).
\end{align*}
\]

Now, observe that \( W_{m+2^i l}(2^{-(i+1)}) = (-1)^i \) and that \( m + 2^i l = m + 2^i l \).

Hence, the above is

\[
\begin{align*}
&= \sum_{i=0}^{2^N-i-1} (-t^{2^i})^i W_{2^i l}(x) \sum_{m=0}^{2^i-1} W_m(x) t^m.
\end{align*}
\]

The proof is completed by noting that \( W_{2^i l}(x) = W_l(2^i x) \). Q.E.D.

**LEMMA 2.** Suppose \( x = \sum_{K=0}^{n} 2^{-i_K} \) and \( N \geq i_n > i_{n-1} > i_0 > 0 \). Then

\[
P_N(t, x) = \frac{1 - t^{2^N}}{1 - t} \prod_{K=0}^{n} \left( 1 - \frac{t^{2^{i_K-1}}}{1 + t^{2^{i_K-1}}} \right).
\]

**PROOF.** Note that if \( x \) is an integer, \( W_K(x) = 0 \) for all \( K \) and, hence, \( P_N(t, x) \)
is a geometric series which sums to $(1 - t^{2^N})/(1 - t)$. Hence, by Lemma 1,

$$P_N(t, 2^{-i_0}) = P_{i_0 - 1}(t, 0) P_{N - i_0 + 1}(-t^{2^{i_0 - 1}}, 2^{i_0 - 1}, 0)$$

$$= \frac{1 - t^{2^{i_0 - 1}}}{1 - t} \frac{1 - (-t^{2^{i_0 - 1}})^{2^{N-i_0 + 1}}}{1 + t^{2^{i_0 - 1}}} = \frac{1 - t^{2^N}}{1 - t} \frac{1 - t^{2^{i_0 - 1}}}{1 + t^{2^{i_0 - 1}}}.$$  

(Note that $2^{N-i_0 + 1}$ is even.) Hence, the formula is true if $n = 0$. Now, assume it true for all integers less than $n$. Let $x_0 = x - 2^{-i_n}$. Then

$$P_N(t, x) = P_N(t, x_0 + 2^{-i_n}) = P_{i_n - 1}(t, x_0) P_{N-i_n+1}(-t^{2^{i_n-1}}, 2^{i_n-1}, x_0).$$

Since $2^{i_n-1}x_0$ is an integer, the $P_{N-i_n+1}$ term equals $(1 - t^{2^N})/(1 + t^{2^{i_n-1}})$. Applying the induction hypothesis to the other term and simplifying yields the result. Q.E.D.

We can now prove formula (1) above. If $x$ is a diadic rational, (1) follows from Lemma 2 by letting $N \to \infty$, so we may suppose that we are using an infinite expansion. Let $x = \sum_{i=0}^{\infty} 2^{-i_k}$ where $0 < i_0 < i_1 < \cdots < i_j < \cdots$, and for each $n \in N$ let $x_n = \sum_{k=0}^{n} 2^{-i_k}$. From the above,

$$P(t, x_n) = \frac{1}{1 - t} \prod_{K=0}^{n} \left( \frac{1 - t^{2^{i_k - 1}}}{1 + t^{2^{i_k - 1}}} \right).$$

Since the series for $P$ converges uniformly in $x$ if $|t| < 1$, $P$ is continuous (in the Walsh sense) in $x$ and, hence, $P(t, x_n) \to P(t, x)$. This proves convergence of the infinite product and formula (1). Q.E.D.

**Corollary 1.** $\tilde{\Omega}(x) \geq 0$ for all $x$ and, hence, $\Omega(x) \geq -1$ for all $x$.

**Proof.** As shown in the introduction, $\tilde{\Omega}(x) = \int_0^1 P(t, x) \, dt$, which is clearly positive. Q.E.D.

**Corollary 2.** There are constants $C_1$ and $C_2$ such that $|\Omega(x)| \leq C_1 \log 1/x + C_2$ for all $x \in (0, 1)$.

**Proof.** If $a > 0$, $(1 - a)/(1 + a) < 1$. Hence,

$$P(t, x) \leq \frac{1}{1 - t} \frac{1 - t^{2^{i_k - 1}}}{1 + t^{2^{i_k - 1}}} \leq \frac{1}{1 - t} (1 - t^{2^{i_k - 1}}) = \sum_{K=0}^{2^{i_k - 1} - 1} t^K.$$

Hence,

$$\tilde{\Omega}(x) = \int_0^1 P(t, x) \, dt \leq \sum_{K=1}^{2^{i_k - 1} - 1} \frac{1}{K} \leq C_1 \log 2^{i_k - 1} + C_2.$$

But $x \leq 2^{i_k - i_0} = 2^{-(i_0 - 1)}$. Hence, $\log 2^{i_k - 1} \leq \log 1/x$, proving the claim for $\tilde{\Omega}$ and, hence, for $\Omega$. Q.E.D.

**Remarks.** Corollary 2 could also be proven analogously to the technique used by Yano [4] to obtain estimates on $\sum W_K/K^\alpha$ ($0 < \alpha < 1$). However, lower bounds do not seem to be so easily obtainable from Yano's technique.

**Corollary 3.** $\Omega$ is continuous on $(0, 1)$, and is unbounded as $x \to 0^+$.
PROOF. Note that if \( x = \sum_{K=0}^{\infty} 2^{-iK} \) as before, then

\[
P(t, x) \leq \frac{1}{1-t} \frac{1-t^{2^{a-1}}}{1+t^{2^{a-1}}} \leq \sum_{K=0}^{2^{a-1}} t^K \leq 2^{b-1} \quad \text{for } t \in (0, 1).
\]

Also note that

\[
\int_{0}^{1} P(t, x) (1-t) \, dt = \sum \left( \frac{1}{n+1} - \frac{1}{n+2} \right) W_{n}(x)
\]

is an absolutely convergent Fourier series and, hence, is Walsh continuous.

Now write \( x_n = \sum_{K=0}^{n} 2^{-iK} \). It follows trivially from (1) that \( P(t, x) = P(t, x - x_n)(1-t)P(t, x_n) \). Hence, if \( y \) is such that \( y_n = x_n \), then

\[
\left| \int_{0}^{1} P(t, x) - P(t, y) \, dt \right| = \left| \int_{0}^{1} \left[ P(t, x - x_m)(1-t) - P(t, y - y_n)(1-t) \right] P(t, x_n) \, dt \right|
\]

\[
\leq 2^{b-1} \left\{ \int_{0}^{1} P(t, x - x_n)(1-t) \, dt + \int_{0}^{1} P(t, y - y_n)(1-t) \, dt \right\}.
\]

As \( n \to \infty \) this tends to zero from the continuity of \( \int_{0}^{1} P(t, x)(1-t) \, dt \). The unboundedness is trivial since

\[
\Omega(2^{-(i+1)}) = \int_{0}^{1} \frac{1-t^{2i}}{(1+t^{2i})(1-t)} \, dt,
\]

which \( \to \infty \) as \( i \to \infty \). Q.E.D.

REMARKS. Ladhawala has constructed a more direct proof of the continuity of \( \Omega \) following the proof that \( \Omega \) is in \( L^1 \) given in [1]. Note, incidentally, that \( \Omega \in L^1 \) follows trivially from \( \Omega \in L^2 \).

Now, one of the basic properties of differentiation is that if \( f \in L^1(\mathbb{R}) \) and \( \lim_{t \to 0} (f(\cdot + t) - f(\cdot))/t \) exists in \( L^1 \), then \( f \) is absolutely continuous. (One simply integrates this limit to obtain \( f \).) The corresponding theorem for Walsh differentiation is false. However, the following is true. The first part was pointed out to us by Ladhawala.

COROLLARY 4. If \( f \in L^1([0, 1]) \) and \( Df \) exists in the \( L^1 \) sense (see [1]) and is in \( L^p \) for some \( P > 1 \), then \( f \) is Walsh continuous. However there exist functions in \( L^1([0, 1]) \), differentiable in the \( L^1 \) sense, which are not continuous in the Walsh sense.

PROOF. By results of [1], \( f = \Omega * Df + f(0) \) (\( * \) in the Walsh sense). Since \( \log(1/x) \) is in \( L^q \) for all \( 1 \leq q < \infty \) and \( L^p \) convolved with \( L^q \) is continuous (\( P \) and \( q \) conjugate exponents), the first claim follows.

To prove the second part, note that since \( \Omega \) is in \( L^2 \), \( \Omega * L^1 \subset L^2 \subset L^1 \). From results of [1], every element of \( \Omega * L^1 \) is differentiable in the \( L^1 \) sense, with derivative in \( L^1 \). Hence, if our theorem is false, convolution by \( \Omega \) maps \( L^1 \) into the space of Walsh-continuous functions on \([0, 1]\). By the closed graph
theorem this mapping would have to be continuous from $L^1$ into the uniform topology on $C([0,1])$. In particular, $f \to \Omega \ast f(0) = \int_0^1 f(x)\Omega(x)\,dx$ is continuous in $L^1$, implying that $\Omega$ is essentially bounded, which is false by Corollary 3. Q.E.D.

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References


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