NUMERICAL RANGE OF A WEIGHTED SHIFT WITH PERIODIC WEIGHTS

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Abstract. Calculation of the numerical range of a weighted shift is reduced to the solution of a polynomial equation when the weights form a periodic sequence, or approach a periodic sequence from below.

Introduction. A weighted shift on \( l^2 \) or \( l^1_+ \) is a linear operator \( S \) defined by 
\[
S e_n = s_n e_{n+1}
\]
where \( \{e_n\} \) is an orthonormal basis, and \( \{s_n\} \) a sequence of complex numbers. The numerical range of an operator \( S \) is the set of complex numbers \( \{Sx, x\} \) where \( \|x\| = 1 \); this is denoted \( W(S) \). For definiteness we assume here a one-sided shift, indexed by positive integers; the proofs and results are the same for a two-sided shift.

Then we are given
\[
x = x_1 e_1 + x_2 e_2 + \cdots, \quad x_i \text{ complex}, \quad \sum |x_i|^2 < \infty;
\]
\[
Sx = s_1 x_1 e_2 + s_2 x_2 e_3 + \cdots.
\]

We begin with some simple facts about weighted shifts [1].
(1) \( S \) is a bounded operator if and only if \( \{s_n\} \) is a bounded sequence, and then \( \|S\| = \sup |s_n| \).
(2) \( S \) is unitarily equivalent to a shift with weights \( t_i \) whenever \( |t_i| = |s_i| \) for all \( i \). In particular, \( S \) is unitarily equivalent to \( cS \) whenever \( |c| = 1 \).
(3) Therefore \( W(S) \) has circular symmetry about 0: \( cW = W \) whenever \( |c| = 1 \).
(4) Since \( W \) is convex, it follows that \( W(S) \) is a disk centered at 0; its radius \( w(S) \) is the numerical radius of \( S \).

It is an easy exercise to find \( W(S) \) in some special cases. For example:
(5) If \( |s_n| \leq K \) for all \( n \), and \( |s_n| \to K \), then \( W(S) = K \).

By (2) it suffices to consider shifts with real nonnegative weights, \( s_n \geq 0 \), and we shall do so.

Theorem 1. If \( \{s_n\} \) is a periodic sequence, of period \( r \), then
\[
w(S) = \max \{s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1 : x_i \text{ real}, \quad x_1^2 + \cdots + x_r^2 = 1\},
\]
and finding this is equivalent to solving a polynomial equation of degree \( r \).
Proof. First consider a sequence $x$ consisting of the finite sequence of complex numbers $\{x_1, x_2, \ldots, x_r\}$ repeated $k$ times, with $0$'s thereafter. Then

$$Sx = \{0, [s_1 x_1, s_2 x_2, \ldots, s_r x_r], \text{(repeated } k \text{ times)}, 0, 0, \ldots\},$$

$$(Sx, x) = k(s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1) - s_r x_r x_1,$$

$$(x, x) = k(|x_1|^2 + |x_2|^2 + \cdots + |x_r|^2),$$

and for large $k$ we see that $(Sx, x)/(x, x)$ can be made arbitrarily close to

$$\frac{s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1}{|x_1|^2 + |x_2|^2 + \cdots + |x_r|^2}.$$

Therefore $w(S)$ is at least equal to

$$\max\{|s_1 x_1 x_2 + \cdots + s_r x_r x_1|: x_i \text{ complex, } |x_1|^2 + \cdots + |x_r|^2 = 1\}.$$

By multiplying $x_k$ by $e^{i\theta_k}$ we may make these components real and nonnegative: this gives the problem:

Maximize $s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1$

subject to $x_1^2 + \cdots + x_r^2 = 1, s_k, x_k$ real.

The use of Lagrange multipliers gives the system:

$$s_r x_r + s_1 x_2 = \lambda x_1$$
$$s_1 x_1 + s_2 x_2 = \lambda x_2$$
$$s_{r-1} x_{r-1} + s_r x_1 = \lambda x_r.$$

Elimination of the $x_i$ gives a polynomial equation in $\lambda$ of degree $r$; $x_2, \ldots, x_r$ are found by substitution (in terms of $x_1$), and $x_1$ is then found by the relation $x_1^2 + \cdots + x_r^2 = 1$.

We now establish that $w(S)$ is no greater than this maximum value of $s_1 x_1 x_2 + \cdots + s_r x_r x_1$.

Lemma. If $a_k, b_k$ are nonnegative constants with $b_k \neq 0$, then

$$\frac{a_1 + a_2 + \cdots}{b_1 + b_2 + \cdots} \leq \sup_k \frac{a_k}{b_k}$$

whenever the left side is defined.

Proof. We first show this for finite sums. If $a/c \geq b/d$, then

$$\frac{a + b}{c + d} \leq \frac{a + ad/c}{c + d} = \frac{a}{c}$$

and the result for finite sums follows by induction. Then

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \max_{k \leq n} \frac{a_k}{b_k} \leq \sup_{k \geq 1} \frac{a_k}{b_k}$$
and hence the lim sup of the left side satisfies the same inequality; this proves the lemma.

Resuming the proof of Theorem 1: suppose \(|x| = 1\), and write the components of \(x\) as

\[x = \{a_{11} a_{12} \cdots a_{1r}; a_{21} a_{22} \cdots a_{2r}; \ldots\},\]

\(a_j\) complex. Then

\[
\frac{|(Sx, x)|}{(x, x)} = \frac{|s_1 a_{11} a_{12} + \cdots + s_r a_{1r} a_{21} + s_1 a_{21} a_{22} + \cdots + s_r a_{2r} a_{31} + \cdots|}{|a_{11}|^2 + \cdots + |a_{tr}|^2 + |a_{21}|^2 + \cdots + |a_{2r}|^2 + \cdots}
\]

(1)

\[
\leq \sup_k \frac{|s_1 a_{k1} a_{k2} + \cdots + s_r a_{kr} a_{(k+1)}|}{|a_{k1}|^2 + \cdots + |a_{kr}|^2}
\]

and

\[
\leq \sup_k \frac{|s_1 a_{k1} a_{k2} + \cdots + s_r a_{kr} a_{(k+1)}|}{|a_{(k+1)}|^2 + |a_{k1}|^2 + \cdots + |a_{kr}|^2}.
\]

Inequality (1) follows from the lemma, and (2) follows by deleting \(|a_{11}|^2\) from the denominator (thus increasing the value of the fraction), regrouping terms of the denominator, and applying the lemma.

Setting

\[x_1 = \max(|a_{k1}|, |a_{(k+1)})|), \quad x_j = |a_{kj}| \quad \text{for } j = 2, \ldots, r,
\]

we see that

\[
\frac{|(Sx, x)|}{(x, x)} \leq \max\{s_1 x_1 x_2 + \cdots + s_r x_r x_1: x_1^2 + \cdots + x_r^2 = 1, x_j \text{ real}\},
\]

and therefore \(w(S)\) is equal to this maximum value; this completes the proof of Theorem 1.

**Theorem 2.** If \(s_k \leq p_k\) and \(s_k - p_k \to 0\) as \(k \to \infty\), where \(\{p_k\}\) is a periodic sequence with period \(r\), then

\[w(S) = \max\{p_1 x_1 x_2 + \cdots + p_r x_r x_1: x_1^2 + \cdots + x_r^2 = 1, x_i \text{ real}\}.
\]

**Proof.** Given \(\epsilon > 0\), letting \(T\) be the shift with weights \(p_k\), let \(x\) be a unit vector such that \((Tx, x) > w(T) - \epsilon\). Choose \(n\) such that \(|s_k - p_k| < \epsilon\) for \(k \geq n\). Let \(y\) be the unit vector with \(y_k = 0, k = 1, \ldots, n; y_{k+n} = x_k, k = 1, 2, \ldots\). Then \((Ty, y) = (Tx, x) > w(T) - \epsilon\).

Now

\[\|Ty - Sy\| \leq \sup_{k > n} |p_k - s_k| \leq \epsilon
\]

so \(|(Ty, y) - (Sy, y)| \leq \epsilon\) and so \((Sy, y) > w(T) - 2\epsilon\).

Therefore \(w(S) \geq w(T)\). Since \(0 \leq s_k \leq p_k\), we easily have \(w(S) \leq w(T)\), and so the two are equal. By Theorem 1,

\[w(T) = \max\{p_1 x_1 x_2 + \cdots + p_r x_r x_1: x_1^2 + \cdots + x_r^2 = 1, x_i \text{ real}\} = w(S);
\]

this proves Theorem 2.
EXAMPLES. (1) If \( r = 2 \) the weights are \( a, b, a, b, \ldots \); we are to maximize \((a + b)x_1x_2\), that is, maximize \( x_1x_2 \) subject to \( x_1^2 + x_2^2 = 1 \). The solution is \( x_1 = x_2 = 1/\sqrt{2} \), \((a + b)x_1x_2 = (a + b)/2\), and so the numerical radius is the average of the two weights.

(2) If \( r = 3 \) the weights are \( a, b, c, a, b, c, \ldots \); we must maximize \( ax_1x_2 + bx_2x_3 + cx_3x_1 \) with \( x_1^2 + \cdots + x_3^2 = 1 \); we have the system

\[
ax_2 + cx_3 = \lambda x_1, \quad ax_1 + bx_3 = \lambda x_2, \quad bx_2 + cx_1 = \lambda x_3,
\]
which (if \( x_1 \neq 0 \)) leads to the cubic equation \( \lambda^3 - (a^2 + b^2 + c^2)\lambda - 2abc = 0 \).

NOTES. (1) The numerical range is always a disk about 0, of positive radius except in the trivial case where all the weights are zero.

(2) If any weight is zero then the disk is closed; for (assuming \( s_r = 0 \) for example) then \((Sx, x)/(x, x)\) is actually equal to the expression (1), which in turn attains its maximum on the compact sphere (2).

(3) For weights \( 1, 1, 1, \ldots \) the disk is open; for \(|(Sx, x)| = 1, |x| = 1\), would imply \( Sx = kx, |k| = 1 \), which is impossible.

(4) I surmise, but have yet to prove, that the disk is open whenever all the weights are nonzero; that is, \((Sx, x)\) cannot attain its sup \( w(S) \) for \(|x| = 1\).

REFERENCE


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