ON THE SPACE OF PIECEWISE LINEAR HOMEOMORPHISMS OF A MANIFOLD

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Abstract. Let $M^n$ be a compact PL manifold, $n \neq 4$; if $n = 5$, suppose $\partial M$ is empty. Let $H(M)$ be the space of homeomorphisms on $M$ and $H^*(M)$ the elements of $H(M)$ which are isotopic to PL homeomorphisms. It is shown that the space of PL homeomorphisms, $PLH(M)$, has the finite dimensional compact absorption property in $H^*(M)$ and hence that $(H^*(M), PLH(M))$ is an $(l_2, l_2^2)$-manifold pair if and only if $H(M)$ is an $l_2$-manifold. In particular, if $M^2$ is a 2-manifold, $(H(M^2), PLH(M^2))$ is an $(l_2, l_2^2)$-manifold pair.

1. Introduction. Let $M$ be a compact piecewise linear (PL) manifold, possibly with boundary. We shall study the pair $(H(M), PLH(M))$, where $H(M)$ denotes the space of all homeomorphisms of $M$ onto itself and $PLH(M)$ the subspace of all piecewise linear homeomorphisms. All function spaces will be assumed to have the compact-open topology.

For some years now there has been considerable interest in the question of whether $H(M)$ is an $l_2$-manifold; i.e., a separable metric space which is locally homeomorphic to $l_2$, the hilbert space of square-summable sequences. For an arbitrary compact manifold $M$, it is known that $H(M)$ is uniformly locally contractible (Chernavskii [6] and Edwards and Kirby [8]) and that $H(M) \times l_2$ is homeomorphic to $H(M)$ (Geoghegan [11]). By a theorem of Toruńczyk [20], $H(M)$ is an $l_2$-manifold if and only if $H(M)$ is an ANR. When $M$ is a 2-manifold, Luke and Mason [18] have shown that $H(M)$ is an ANR (hence an $l_2$-manifold). But it is still unknown whether or not $H(M^n)$ is an ANR when $n > 2$.

In the PL case more is known. $PLH(M)$ is the countable union of finite-dimensional compacta (Geoghegan [12]), and is uniformly locally contractible (Edwards, see [13] and Gauld [10]). Haver [13] has shown that any such space is an ANR. Toruńczyk [20] has shown that any such ANR becomes an $l_2^2$-manifold when multiplied by $l_2^2$ (where $l_2^2$ denotes the subspace of $l_2$ consisting of those sequences having only finitely many nonzero entries). Hence $PLH(M) \times l_2^2$ is an $l_2^2$-manifold. Finally, Keesling and Wilson [16] have shown that $PLH(M) \times l_2^2$ is homeomorphic to $PLH(M)$. Hence $PLH(M)$ is an $l_2^2$-manifold.

In general it is not true that $PLH(M)$ is dense in $H(M)$. For example, Kirby and Siebenmann (see [17]) have shown that $H(S^2 \times S^3)$ has a component containing no PL homeomorphism. They have also shown that if the

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third cohomology group of $M^n$ with $\mathbb{Z}_2$ -coefficients is trivial, and if $n \geq 6$, or $n = 5$ and $\partial M$ is empty, $PLH(M^n)$ is dense in $H(M^n)$. In low dimensions, $PLH(M^n)$ is always dense in $H(M^n)$: this is obvious when $n = 1$; see Rado [19] when $n = 2$; see Bing [4] when $n = 3$.

For our purposes we can avoid reference to the difficult work of Kirby and Siebenmann. Using earlier work of Connell [7] we can prove

**Theorem 1.** Let $M^n$ be a compact PL manifold, $n \neq 4$; if $n = 5$ suppose $\partial M$ is empty. Then the closure of $PLH(M^n)$ in $H(M^n)$ is the union of components of $H(M^n)$.

Let $H^\ast(M^n)$ be the subset of $H(M^n)$ consisting of those homeomorphisms which are isotopic to PL homeomorphisms. The above remarks show that $H^\ast(M^n)$ is often equal to $H(M^n)$. Theorem 1 says that [with certain dimension restrictions] $PLH(M^n)$ is always dense in $H^\ast(M^n)$.

A pair of spaces $(X, X')$ is an $(l_2, l^1)$-manifold pair if $X$ is an $l_2$-manifold, and if there exist an open cover $\mathcal{U}$ of $X$ and open embeddings $\{f_U : U \to l_2 | U \in \mathcal{U}\}$ such that for each $U, f_U(U \cap X') = f_U(U) \cap l^1$. In other words $X'$ sits in $X$, locally, as $l^1$ sits in $l_2$.

**Theorem 2.** Let $M^n$ be a compact PL manifold, $n \neq 4$; if $n = 5$, suppose $\partial M$ is empty. Then $(H^\ast(M^n), PLH(M^n))$ is an $(l_2, l^1)$-manifold pair if and only if $H(M^n)$ is an $l_2$-manifold.

Theorem 2 is proved by showing that $PLH(M^n)$ has the “finite-dimensional compact absorption property” in $H^\ast(M^n)$; see §3 for the definition. This theorem, when combined with our opening remarks, yields

**Corollary 1.** If $M^2$ is a compact PL 2-manifold, $(H(M^2), PLH(M^2))$ is an $(l_2, l^1)$-manifold pair.

Theorem 1 combined with the fact that each of $PLH(M^n)$ and $H(M^n)$ is uniformly locally contractible immediately gives:

**Corollary 2.** Let $M^n$ be as in Theorem 2. Then the inclusion $PLH(M^n) \to H^\ast(M^n)$ is a weak homotopy equivalence (a homotopy equivalence if $n = 2$).

**Corollary 3.** The inclusion of the identity component of $PLH(M^n)$ into the identity component of $H(M^n)$ is a weak homotopy equivalence (a homotopy equivalence if $n = 2$).

We will let $H_3(I^n)$ denote the set of elements of $H(I^n)$ which equal the identity when restricted to the boundary of $I^n$ and $PLH_0(I^n) = PLH(I^n) \cap H_0(I^n)$. If $X$ is any space, let $l_X$ denote the identity homeomorphism on $X$. If $f \in H(M^n)$, let $N_\epsilon(f) = \{ h \in H(M^n) | d(h, f) < \epsilon \}$ where $d$ is a metric on $M$.

2. **Proof of Theorem 1.** As explained in §1, the low-dimensional cases are well known.

**Lemma 1.** If $n > 4$, $PLH_0(I^n)$ is dense in $H_0(I^n)$.

**Proof.** Let $H_3'(I^n)$ be the group of homeomorphisms of $I^n$ which fix a neighborhood of $\partial I^n$. More precisely,
H_δ'^{(I^n)}(I^n) = \{ h \in H_δ'(I^n) | \text{for some } 0 < a < 1, \\
h(x) = x \text{ whenever } d(x,0) \geq a \}.

By [9, Theorem 3], H_δ'^{(I^n)}(I^n) is a simple subgroup of H_δ(I^n). H_δ'^{(I^n)}(I^n) is clearly dense in H_δ(I^n). Consider

H_δ''(I^n) = \{ h \in H_δ'(I^n) | \text{If } T \text{ is a PL structure on } I^n \text{ and if } \epsilon > 0, \\
\text{there exists } f \in H_δ'(I^n) \text{ such that} \\
f \text{ is PL relative to } T \text{ and } d(f,h) < \epsilon \}.

\text{Claim (following Connell). } H_δ''(I^n) \text{ is normal in } H_δ'(I^n).

\text{Proof of Claim.} \text{ Let } h \in H_δ''(I^n) \text{ and } g \in H_δ'(I^n). \text{ We must show } g^{-1}hg \in H_δ''(I^n). \text{ Let } T \text{ be a PL structure on } I^n \text{ and let } \epsilon > 0 \text{ be given. There exists } \delta > 0 \text{ such that for all } x, y \in I^n, d(x,y) < \delta \text{ implies } d(g^{-1}(x),g^{-1}(y)) < \epsilon. \\
\text{Let } T_0 \text{ be the PL structure on } I^n \text{ which is the image of } T \text{ under } g. \text{ Since } h \in H_δ''(I^n) \text{ there exists } f \in H_δ'(I^n) \text{ which is PL with respect to } T_0 \text{ and satisfies } d(h,f) < \delta. \text{ Thus } d(g^{-1}hg,g^{-1}fg) < \epsilon. \text{ Now } g^{-1}fg \text{ is PL with respect} \text{ to } T. \text{ It follows that } g^{-1}hg \in H_δ''(I^n); \text{ the Claim is proved.}

\text{Since } H_δ'(I^n) \text{ is dense and simple, it only remains to show that } H_δ''(I^n) \text{ contains some } h_0 \neq 1_{I^n}. \text{ When } n > 6, \text{ Connell shows on p. 331 of [7] that any nontrivial symmetric radial expansion of } I^n \text{ which fixes a neighborhood of } \partial I^n \text{ can serve as } h_0. \text{ The restriction } n > 6 \text{ arises from the radial engulfing lemma used in [7]. But in [3], Bing proves a stronger engulfing lemma which makes Connell's proof valid for } n > 4; \text{ see [3, p. 3] and [7, p. 337].}

\text{Proof of Theorem 1.} \text{ Let } H \text{ be a component of } H(M). \text{ We show that} \\
H \cap PLH(M) \text{ is either dense in } H \text{ or is empty.}

\text{We first show the PL homeomorphisms are dense in a neighborhood of } 1_M: \text{ i.e., that there is a } \delta > 0 \text{ such that whenever } g \in N_\delta(1_M) \text{ and } \epsilon > 0 \text{ are given, there exists } h \in PLH(M) \cap N_\epsilon(g). \\
\text{Choose an open cover } \{ B_1, \ldots, B_p \} \text{ of } M \text{ so that for each } i, \partial B_i \text{ is a PL } n\text{-ball and so that } D_i = \overline{B_i} \cap \partial M \text{ is either empty or a PL } (n-1)\text{-ball. In [8, p. 19] it is shown that there is a } \delta > 0 \text{ so that each } g \in N_\delta(1_M) \text{ can be written as the composition} \\
g = g_pg_{p-1} \cdots g_0 \text{ of } p \text{ homeomorphisms such that each } g_i \text{ is supported by } B_i. \text{ Since each } g_i \text{ is uniformly continuous, there exist positive numbers } \eta_1, \ldots, \eta_p \text{ such that if for each } i \text{ the homeomorphism } h_i \text{ satisfies} \\
d(h_i,g_i) < \eta_i, \text{ then } d(h_i,\overline{B_i}) < \eta_i, \text{ and } d(h_i,\overline{B_i}) < \eta_i. \text{ If } \partial \overline{B_i} \text{ maps } D_i \text{ homeomorphically onto itself and fixes } \partial D_i. \text{ By the previous lemma there is a PL homeomorphism } f_i \in H_\eta(D_i) \text{ approximating } g_i | D_i. \text{ Extending } f_i \text{ by the identity we get a PL homeomorphism } f_i^i: \partial \overline{B_i} \rightarrow \partial \overline{B_i} \text{ which approximates } g_i | \partial \overline{B_i}. \\
\text{Thus } f_ig_i^{-1} | \partial \overline{B_i} \text{ approximates } 1 \overline{B_i}. \text{ Coning at a point in the interior of } B_i, \text{ we obtain a homeomorphism } F_i \text{ of } \overline{B_i} \text{ which agrees with } f_ig_i^{-1} \text{ on } \partial \overline{B_i} \text{ and which approximates } 1_{\overline{B_i}}. \text{ Let } g_i' = F_i g_i: \overline{B_i} \rightarrow \overline{B_i}. \text{ Note that } g_i' \text{ approximates } g_i \text{ and is PL on } \partial \overline{B_i}. \text{ Omitting epsilonics, we may say that } d(g_i',g_i | B_i) < \eta_i/2. \text{ Note}
that $g_i'$ fixes $\partial B_i \cap \text{int } D_i$. By coning, construct a PL homeomorphism $g_i'' : \overline{B}_i \to \overline{B}_i$ which extends $g_i' | \partial B_i$. Then $g_i''(g_i'')^{-1} \in H_3(B_i)$, so there exists $\tilde{g}_i \in PLH_3(B_i)$ such that $d(\tilde{g}_i, g_i''(g_i'')^{-1}) < \eta_i/2$. Extend $\tilde{g}_i g_i''$ by the identity to $h_i \in PLH(M)$ and note that $d(h_i, g_i) < \eta_i$. The required $h$ is $h_p \cdots h_1$.

Now let $H$ be a component of $H(M)$ such that $H \cap PLH(M) \neq \emptyset$. Let $A = \text{cl}_H (H \cap PLH(M))$. We will show that $A$ is open in $H$, from which it will follow that $A = H$. Let $f \in A$. Let $\delta$ be as in the first part of the proof, and assume without loss of generality that $N_{2\delta}(f) \subset H$. The local connectedness of $H(M)$ is used here. Let $g_0 \in N_{\delta}(f)$. We must show that $g_0 \in A$. Let $\eta > 0$ be given: we may assume $\eta \leq \delta$. Since $d(g_0 f^{-1}, 1_M) < \delta$, there exists $h \in PLH(M)$ such that $d(h, g_0 f^{-1}) < \eta/2$ and hence $d(hf, g_0) < \eta/2$. Since $h$ is uniformly continuous, there exists $\gamma > 0$ such that $d(h(x), h(y)) < \eta/2$ whenever $d(x, y) < \gamma$. Choose $f' \in PLH(M)$ such that $d(f', f) < \gamma$. Then $hf' \in PLH(M)$ and $d(hf', g_0) \leq d(hf', hf) + d(hf, g_0) < \eta$. Therefore $hf' \in N_\eta(g_0) \cap H$. Since $hf'$ is PL, $g_0 \in A$.

3. Proof of Theorem 2. A subset $X'$ of a metric space $X$ is said to have the finite-dimensional compact absorption property (f.d. cap) if $X' = \bigcup_{n=1}^{\infty} A_n$ where (i) each $A_n$ is finite dimensional and compact, and (ii) given a finite-dimensional compactum $A$, a closed subset of $B$ of $A$, an embedding $f : A \to X$ such that $f(B) \subset X'$, and a number $\epsilon > 0$, there exists an embedding $h : A \to X$ such that $d(f, h) < \epsilon$, $h(A) \subset X'$ and $h(b) = f(b)$ whenever $b \in B$.

This property characterizes $(l_2, l_2^1)$-manifold pairs: precisely

**Proposition 1 (West [21]).** Let $(X, X')$ be a pair of metric spaces such that $X'$ has the f.d. cap in $X$. Then $(X, X')$ is an $(l_2, l_2^1)$-manifold pair if and only if $X$ is an $l_2$-manifold.

Where there is no confusion, we will not distinguish between a complex $P$, and its underlying point set, $|P|$. Let $\overline{PLH}(M)$ be the closure of $PLH(M)$ in $H(M)$.

**Lemma 2.** Let $P$ be a finite complex and $B$ a closed subset of $P$. Suppose $f : P \to \overline{PLH}(M)$ is a map such that $f(B) \subset PLH(M)$ and $f(P \setminus B) \cap f(B) = \emptyset$. Then given $\epsilon > 0$, there is a map $g : P \to PLH(M)$ such that $d(f, g) < \epsilon$, $g|B = f|B$ and $g(P \setminus B) \cap g(B) = \emptyset$.

**Proof.** Let $n = \dim P$. Whenever $W$ is a closed subset of $P \setminus B$ let $\eta_W = 1/4 \min(\epsilon, d(f(W), f(B)))$.

Using the fact that $PLH(M)$ is uniformly locally contractible, form a triangulation $T$ of $P \setminus B$ satisfying:

(a) for each simplex $\sigma$ of $T$ there is a sequence $0 < \delta_{\sigma,0} < \delta_{\sigma,1} < \cdots < \delta_{\sigma,n} \leq \eta_\sigma$ such that any subset of $PLH(M)$ of diameter $< 3\delta_{\sigma,i}$ is contractible in a subset of $PLH(M)$ of diameter $< \delta_{\sigma,i+1}, 0 \leq i < n - 1$; and

(b) diam $f(\sigma) < \delta_{\sigma,0}$. (First choose a triangulation $\hat{T}$, next choose the $\delta_{\sigma,i}$'s with reference to $\hat{T}$ to satisfy (a); then subdivide $\hat{T}$ to get $T$ satisfying (b).)

Define $g : T^0 \to PLH(M)$ as follows: If $v$ is a vertex of $T^0$, let $g(v)$ be any
point of $\text{PLH}(M)$ such that $d(g(\nu), f(\nu)) < \delta_{\nu,0}$ for each of the (finite number of) simplexes $\sigma$ of $T$ which contains $\nu$ as a vertex.

We next define $g : T^1 \to \text{PLH}(M)$. Let $\tau$ be a 1-simplex of $T^1$ and let $\sigma_0 \subset T$ be a simplex of which $\tau$ is a face. Then $\text{diam} f(\tau) \leq \text{diam} f(\sigma_0)$. Hence $\text{diam} g(\partial \tau) < 3\delta_{\sigma_0,0}$. We can therefore define

$$g|_{\partial \tau} : \tau \to \text{PLH}(M)$$

extending $g|_{\partial \tau}$ in such a way that $\text{diam} g(\tau) < \delta_{\sigma,1}$ for each of the (finite number of) simplexes $\sigma$ of $T$ which contains $\tau$ as a face.

Assume, inductively, that we have defined $g|T^j : T^j \to \text{PLH}(M)$, such that if $\tau$ is a $j$-simplex of $T$, then $\text{diam} g(\tau) < \delta_{\sigma, j}$ for each of the (finite number of) simplexes $\sigma$ of $T$ which contains $\tau$ as a face.

Let $\tau$ be a $(j + 1)$-simplex of $T$. Note that $\text{diam} g(\partial \tau) < 3\delta_{\sigma, j}$ for each simplex $\sigma$ of $T$ which contains $\tau$ as a face. Therefore $g|_{\partial \tau}$ can be extended to $\tau$ in such a way that $\text{diam} g(\tau) < \delta_{\sigma, j+1}$ for each simplex $\sigma$ of $T$ of which $\tau$ is a face.

Thus, by induction we define a continuous map $g : T \to \text{PLH}(M)$ so that if $\tau$ is a simplex of $T$, $\text{diam} g(\tau) < \eta \tau$. Finally define $g$ on $B$ to agree with $f$.

We now show that $d(f(\nu), g(\nu)) < \epsilon$. If $p \in B$, $d(f(p), g(p)) = 0$. If $p \in P \setminus B$ let $\tau$ be a simplex of $T$ containing $p$: $d(f(p), g(p)) \leq 3\eta \tau < \epsilon$.

To show that $g(P \setminus B) \cap g(B) = \emptyset$, we observe that if $p \in P \setminus B$ and $p$ lies in the simplex $\tau$ of $T$,

$$d(g(p), g(B)) \geq d(f(p), f(B)) - d(f(p), g(p))$$

$$\geq d(f(p), f(B)) - 3\eta \tau > 0.$$ 

Finally we show that $g$ is continuous on $B$. Let $g \in B$ and $\eta > 0$ be given. Choose $\delta > 0$ so that if $d(\nu, q) < \delta$, $d(f(\nu), f(q)) < \eta/2$. Assume $p \in P \setminus B$. Then

$$d(g(p), g(q)) \leq d(g(p), f(p)) + d(f(p), f(q))$$

$$\leq d(f(\tau), f(B)) + d(f(p), f(q)) \leq 2d(f(p), f(q)) < \eta.$$ 

Let $s$ denote the countable infinite product of open unit intervals and $s_j = \{[x_i] \in s \mid$ for at most finitely many $i, x_i \neq 0\}$ with metric $d(x,y) = \sum_i \frac{1}{2^i} |x_i - y_i|$. 

**Lemma 3.** Let $P$ be a finite complex, $B$ a closed subset of $P$, $V$ an open subset of $s_f$. Let $g : P \to V$ be a continuous map. Then given $\epsilon > 0$, there is a map $h : P \to V$ such that $h|B = g|B$, $d(g, h) < \epsilon$, $h(B) \cap h(P \setminus B) = \emptyset$ and $h|P \setminus B$ is injective.

**Proof.** This is standard infinite-dimensional topology: we sketch the proof, leaving epsilonics to the reader. Let $p_m : s_f \to s_f$ be the projection onto the first $m$ coordinates. If $m$ is large, $g$ is homotopic in $V$ to $p_m g$ by a small homotopy. If $m$ is large enough, $p_m g$ is homotopic in $V$ to an embedding $h' : P \to V$ by a small homotopy. $h' g^{-1}|g(B)$ is a homeomorphism of $g(B)$ onto $h'(B)$ which is homotopic to the identity by a small homotopy. By Theorem 2.25 of [5], there
is a homeomorphism \( k \) of \( s_x \) close to the identity, extending \( h'g^{-1}|g(B) \). The required \( h \) is \( k^{-1}h' : P \rightarrow V \).

**Proof of Theorem 2.** By Theorem 1, \( H^*(M) \) is an \( l_2 \)-manifold if and only if \( H(M) \) is. By Proposition 1 it is enough to show that \( PLH(M) \) has the f.d. cap in \( H^*(M) \). By [12, Theorem 1.9], \( PLH(M) \) is the countable union of finite-dimensional compacta. We must verify (ii) the f.d. cap.

Suppose we are given a finite-dimensional compactum \( A \), a compact subset \( B \), an embedding \( f : A \rightarrow H^*(M) \) such that \( f(B) \subset PLH(M) \), and \( \epsilon > 0 \). By Theorem 1, \( PLH(M) = H^*(M) \). Since \( H^*(M) \) is locally contractible, \( f \) extends to a map \( f' : P \rightarrow H^*(M) \) where \( P \) is a finite complex (see [15, p.150]).

As explained in §1, \( PLH(M) \) is homeomorphic to an open subset of \( l_2^f \), and hence to an open subset of \( s_x \) since \( l_2^f \) and \( s_x \) are homeomorphic [1], [2]. Thus, Lemmas 2 and 3 give a map \( h : P \rightarrow PLH(M) \) with \( d(f', h) < \epsilon \), \( hB' = f'^{-1}(B') \) where \( B' = f'^{-1}(f(B)) \), \( h(B') \cap h(P \setminus B') = \emptyset \) and \( h|P \setminus B' \) is injective. \( h|A \) is the desired embedding.

**References**


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