ON REALIZING CENTRALIZERS OF CERTAIN ELEMENTS IN
THE FUNDAMENTAL GROUP OF A 3-MANIFOLD

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Abstract. The main result in this note is that if \( \lambda \) is a simple loop in the
boundary of a compact, irreducible, orientable 3-manifold \( M \) and \( [\lambda] \neq 1 \in \pi_1(M) \), one can represent the centralizer of \([\lambda]\) in \( \pi_1(M) \) by a Seifert fibred
submanifold of \( M \).

Introduction. The main result in this note is a partial answer to a question
of Jaco [2]. Jaco has shown in [2] that the centralizer of a nontrivial element
in the fundamental group of a sufficiently large compact, orientable 3-
manifold is isomorphic to the fundamental group of a Seifert fibre space. He
also suggests that one might geometrically realize this group by a submanifold
of the ambient manifold. It is the purpose of this note to show that the above
realization can in fact be made if the element is represented by a simple loop
in the boundary of the 3-manifold and the 3-manifold is irreducible.

Proposition 7.1 in [2] is quite similar to our Theorem 2. Our notation and
definitions are standard unless otherwise indicated.

We say that a manifold \( N \) is properly embedded in a manifold \( M \) if
\( N \cap \partial M = \partial N \). Let \( A \) be an annulus. A spanning arc \( \alpha \) of \( A \) is an arc properly
embedded in \( A \) such that \( A - \alpha \) is simply connected. Throughout the
remainder of this paper \( \alpha \) will denote a spanning arc of \( A \).

Proposition 1. Let \( A_1, \ldots, A_m \) be a collection of annuli properly embedded
in \( M \) such that \( A_i \cap A_j = \partial A_i = \partial A_j \) for \( 1 \leq j < i \leq m \). Let \( f: (A, \partial A) \rightarrow (M, \partial M) \) be a map such that (1) \( f(\partial A) = \partial A_1 \), (2) \( f_\#: \pi_1(A) \rightarrow \pi_1(M) \) is monic, and
(3) \( f(\alpha) \) is not homotopic rel its boundary to an arc in \( \bigcup_{i=1}^{m} A_i \). Then there is an
embedding \( g: (A, \partial A) \rightarrow (M, \partial M) \) such that
(1) \( g(A) \cap A_i = \partial A_i \) for \( i = 1, \ldots, m \),
(2) \( g_\#: \pi_1(A) \rightarrow \pi_1(M) \) is monic,
(3) \( g(\alpha) \) is not homotopic rel its boundary to an arc in \( \bigcup_{i=1}^{m} A_i \).

Proof. We show first that \( f|\partial A \) may be assumed to be an embedding. Let
\( C_1 \) and \( C_2 \) be the components of \( \partial A \). Let \( (\tilde{M}, \rho) \) be the covering space of \( M \)
associated with the subgroup of \( \pi_1(M) \) generated by the class of the simple
loop \( f(C_1) \). Since \( f_\# \pi_1(A) \subseteq \rho_\# \pi_1(M) \), there is a map \( \tilde{f}: (A, \partial A) \rightarrow (\tilde{M}, \partial \tilde{M}) \)
such that \( \rho \tilde{f} = f \). It is a consequence of the theorem in [8] that there is an
embedding \( \tilde{f}_\#: (A, \partial A) \rightarrow (\tilde{M}, \partial \tilde{M}) \) such that \( \tilde{f}_\#(\partial A) = f(\partial A) \). We may

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suppose that \( \tilde{f}(\partial \alpha) = \tilde{f}(\partial \alpha) \). Since \( \pi_1(M) \) is generated by \( \tilde{f}[C_1] \), we may suppose that the loop formed by traversing first \( \tilde{f}(\alpha) \) and then \( f(\alpha) \) is nullhomotopic.

Since the path \( f(\alpha) \) is end-point fixed homotopic in \( M \) to the path \( \rho f(\alpha) \), one may assume that \( f[C_1] \) is an embedding of \( C_1 \) into \( \partial A_1 \). Using a similar argument, we may show that there is no loss in generality in assuming further that \( f[C_2] \) is also an embedding of \( C_2 \) into \( \partial A_1 \).

It is a consequence of Theorem 1' in [1] that there is an embedding \( g: (A, \partial A) \to (M, \partial M) \) such that \( g(\partial A) = \partial A_1 \) and \( g(\alpha) \) is not homotopic rel its boundary to an arc in \( \bigcup_{i=1}^n A_i \). After the usual general position and cutting arguments, we may suppose that \( g^{-1}(\bigcup_{i=1}^n A_i) \) is the union of a collection of disjoint simple essential loops \( \lambda_1, \ldots, \lambda_r \). Let \( A_1, \ldots, A_{r-1} \) be the closures of the components of \( A = \bigcup_{i=1}^n A_i \). We may suppose that \( \alpha \) meets each of the \( \overline{A_i} \) in a spanning arc \( \alpha_i \) for \( i = 1, \ldots, r - 1 \). Since \( g(\alpha) \) is not homotopic rel its boundary to an arc on \( \bigcup_{i=1}^n A_i \), there is a \( j \) such that \( g(\alpha_j) \) is not homotopic rel its boundary to an arc on \( \bigcup_{i=1}^n A_i \). Now it is easy to obtain the desired embedding by considering \( g|\overline{A_j} \). This completes the proof of Proposition 1.

Let \( G \) be a group and \( \sigma \in G \). We denote the centralizer of \( \sigma \) in \( G \), i.e. \( \{ g \in G : \sigma g = g \sigma \} \) by \( C(\sigma) \). The following theorem gives a positive partial answer to a question posed by Jaco in [2].

**Theorem 1.** Let \( M \) be a compact, connected, irreducible, orientable 3-manifold, \( \lambda \) a simple loop in \( \partial M \) which is not nullhomotopic in \( M \), and \( y \) a point in \( \lambda \). Then there is a submanifold \( N \subset M \) such that

1. \( \pi_1((N, y)) = C([\lambda]) \subset \pi_1(M, y) \);
2. \( N \) is a (possibly trivial) Seifert fibre space;
3. \( \partial N \cap \partial M \) is an annular neighborhood \( A^* \) of \( \lambda \);
4. \( \partial N - A^* \) is incompressible in \( M \).

**Proof.** It is a consequence of the loop theorem [4], [6] that \([\lambda]\) is of infinite order in \( \pi_1(M, y) \). Let \( \overline{A} \) be an annular neighborhood of \( \lambda \) in \( \partial M \). Let \( A_1, \ldots, A_n \) be a maximal collection of annuli properly embedded in \( M \) such that

1. \( \partial A_i = \partial \overline{A} \) for \( i = 1, \ldots, n \);
2. \( A_i \cap A_j = \partial A_i \) for \( 1 \leq i < j \leq n \);
3. If \( \alpha_j \) is a spanning arc of \( A_j \) where \( 1 \leq j \leq n \), \( \alpha_j \) is not homotopic rel its boundary to an arc in \( \bigcup_{i \neq j} A_i \cup \overline{A} \).

It follows from the theorem on p. 60 in [7] that there are at most finitely many disjoint nonparallel annuli properly embedded in \( M \) and satisfying condition (1) above.

Suppose \( x \in C([\lambda]) \). Then we claim \( x \) has a representative loop on \( \bigcup_{i=1}^n A_i \cup \overline{A} \).

If our claim is false, we let \( T \) be a torus and \( \lambda_1 \) and \( \lambda_2 \) simple loops on \( T \) such that \( \lambda_1 \cap \lambda_2 \) is a single point and \( T - (\lambda_1 \cup \lambda_2) \) is simply connected. Since \( x \) and \([\lambda]\) commute, there is a map \( \tilde{f}: T \to M \) carrying a neighborhood \( R \) of \( \lambda_1 \) homeomorphically onto \( \overline{A} \) and \( \lambda_2 \) to a representative of \( x \) in \( \pi_1(M) \).

One obtains an annulus \( A \) by removing the interior of \( R \) from \( T \) so that \( \tilde{f} \) induces a map \( f: (A, \partial A) \to (M, \partial A) \). We observe that if \( \alpha \) is a spanning arc of \( A \), \( f(\alpha) \) is not homotopic rel its boundary to an arc on \( \bigcup_{i=1}^n A_i \cup \overline{A} \), since \( x \)
has no representative in that set. Thus it follows from Proposition 1 that the collection was not chosen to be maximal. Our claim follows.

Let $X = \tilde{A} \cup \bigcup_{i=1}^{\infty} A_i$. We claim that $\pi_i(X)$ is the direct product of a free group with the integers. This can be seen by

1. choosing a spanning arc $\alpha$ of $\tilde{A}$,
2. choosing a spanning arc $\alpha_i$ of $A_i$ for $i = 1, \ldots, n$ so $\partial \alpha_i = \alpha \cap A_i$,
3. letting $X_0 = \alpha \cup \bigcup_{i=1}^{n} \alpha_i$,
4. observing that $X$ is naturally homeomorphic to $X_0 \times S^1$.

Let $N_i$ be a regular neighborhood of $X$ in $M$. Note that by construction $N_i$ is homeomorphic to the product of a surface $F$ with $S^1$ and that $\lambda$ is the product of a point with $S^1$ in this structure. Note each component of $\partial N_i$ is a torus. Suppose $T$ is a component of $\partial N_i - \tilde{A}$ such that $\ker(\pi_i(T) \to \pi_i(M)) \neq 1$. Then since $M$ is irreducible, it is a consequence of the loop theorem [4], [6] that $T$ bounds a solid torus in $M$. Let $N$ be the union of $N_i$ with all such solid tori. Now $N$ is a Seifert fibre space and $\ker(\pi_i(N) \to \pi_i(M)) = 1$. This completes the proof of Theorem 1.

**Theorem 2.** Let $M$ be a compact, irreducible, orientable 3-manifold with incompressible boundary and $\lambda$ a simple loop in $\partial M$ such that $[\lambda] \neq 1 \in \pi_i(M)$. Then if there exist integers $n > k > 1$ and $x \in \pi_i(M)$ such that $x^n = [\lambda]^k$, there is a solid torus $N$ embedded in $M$ such that $N \cap \partial M$ is an annular neighborhood of $\lambda$ and $\sigma \in \pi_i(N)$ where $\sigma^n = [\lambda]^k$.

**Proof.** It is a consequence of the loop theorem [4], [6] that $[\lambda]$ is of infinite order. Let $M^*$ be the Seifert fibre space whose existence is guaranteed by Theorem 1. We suppose that $M^*$ is the union of $F \times S^1$ with a collection of solid tori (regular neighborhoods of the exceptional fibres) $N_1, \ldots, N_k$ in the interior of $M^*$, that there is a point $y \in F$ such that $\lambda = \{y\} \times S^1$, and $N_i \cap F \times S^1 = \partial N_i$ for $i = 1, \ldots, k$. Let $\alpha_i$ be a simple arc properly embedded in $F$ such that $\partial (\alpha_i \times S^1)$ is the union of $\lambda$ and a simple loop in $\partial N_i$ for $i = 1, \ldots, k$. Let $R_i$ be a regular neighborhood of $\alpha_i \times S^1 \cup N_i$ and let $M_1$ be the closure of $M^* - R$. Note that $R_1$ is a solid torus.

By van Kampen's theorem $\pi_1(M^*)$ is the free product of $\pi_1(M_1)$ and $\pi_1(R_1)$ with amalgamation over $\pi_1(M_1 \cap R_1) = \langle [\lambda] \rangle$. Note that any conjugate of $x$ in $\pi_1(M^*)$ is an $n$th root of $[\lambda]^k$ since $[\lambda]$ commutes with all elements of $\pi_1(M^*)$. Now it is a consequence of Lemma 10 in [5] that we may suppose that $x$ has length 1 since $\text{length}([\lambda]^k) = 1$ and $\text{length}([\lambda]^k) = \text{length} x^n = n \cdot \text{length} x$ if (length $x$) > 1 where $x$ is assumed to be an element represented by a cyclically reduced word. Thus we may suppose that $x \in \pi_1(R_1)$ or $x \in \pi_1(M_1)$.

If $x \in \pi_1(R_1)$, we are finished. Otherwise we let $M_1^*$ be the closure of $M^* - N_1$. Now $x^n = [\lambda]^k$ holds in $\pi_1(M_1^*)$ since it holds in $\pi_1(M_1)$. Furthermore $M_1^*$ has one less singular fibre than $M^*$. Since the relation $x^n = [\lambda]^k$ holds in $\pi_1(F \times S^1)$ only if $n$ divides $k$, the desired result follows by induction on the number of singular fibres.

In [5] Shalen points out that he has left open the following question: Does every encrusted singular curve have a crust homeomorphic to $S^2 \times S^1$? Since an "encrusted curve" is special and any "special conjugacy class" can be represented by a power of a simple loop by Proposition 2 in [5], one is...
easily able to answer the above question in the affirmative using Theorem 2 above. Jaco has also given an answer to this question in [2].

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