ABSOLUTELY STABLE GAMES

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Abstract. Absolutely stable games, in which every monotone chain of domination reduces to direct domination, are explicitly characterized. Simple games, and n-person games in which all minimal-vital coalitions contain at least n - 1 players, are seen to satisfy the characterization.

Harsanyi [1, pp. 1477-1479] has considered games for which all von Neumann-Morgenstern solutions exhibit a strong form of internal stability. The class of such games includes all absolutely stable games, which Harsanyi defined by a property of chains of domination. It is our purpose to give an explicit characterization of these absolutely stable games.

Let v be a (0, l)-normalized monotonic game, with player set N. A coalition T is minimal-vital if v(T) > 0 and if for every S G, T, v(S) = 0. A monotone chain is a sequence of imputations and coalitions (x0, S1, x1, . . . , Sm,xm) satisfying

1) xk_x dom(Sk) xk, and
2) (x0), > (xj,) for all i G Sk, for 1 < k < m. The game is absolutely stable if for every monotone chain (x0, S1, x1, . . . , Sm,xm), it follows that x0 dominates xm.

Theorem. A game v is absolutely stable if and only if
(a) for every minimal-vital coalition T, if N D S G T then v(S) = v(T), and
(b) for every minimal-vital coalition T with v(T) < 1, every other coalition S with v(S) > 0 satisfies either S G T or S G N - T.

Proof. Sufficiency. Assume that the conditions are satisfied, and consider a monotone chain (x0, S1, x1, . . . , Sm,xm). We will show that x0 dom(Sm) xm.

By (a), all the coalitions of the chain can be assumed to be minimal-vital. By (2), (x0), > (xm), for all i G Sm. To begin, notice that xm_x(Sm) < v(Sm), where for notational convenience we generally write x(S) = 2,xS(x). Proceeding by induction, we assume xk(Sm) < v(Sm) and consider xk_x. If v(Sm) = 1, then

xk_x(Sm) ≤ xk_x(N) = 1 = v(Sm).

Otherwise (b) applies to Sm, and either (i) Sk D Sm, or (ii) Sk G N - Sm.

If (i) applies, then since Sk is minimal-vital, it follows that Sk = Sm and therefore

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$x_{k-1}(S_m) = x_{k-1}(S_k) \leq v(S_k) = v(S_m)$.

On the other hand, if (ii) applies, then $x_{k-1}(N - S_m) > x_k(N - S_m)$, and therefore

$$x_{k-1}(S_m) < x_k(S_m) \leq v(S_m),$$

the last inequality following from the induction hypothesis. Thus in any case it eventually follows that $x_q(S_m) \leq v(S_m)$, and therefore $x_0 \text{dom}(S_m) x_m$.

**Necessity.** Assume that $T$ is minimal-vital with $v(T) < 1$, and consider any $S$ for which $v(S) > 0$, $S \not\supseteq T$, and $S \not\subseteq N - T$. Then $R = N - (S \cup T) \not= \emptyset$ and $T - S \not= \emptyset$. We write $n, r, s, t$ for the respective cardinalities of $N$, $R, S, T$. Let

$$x_i = \begin{cases} (1 - (2s + r)e)/ (t - s) & \text{if } i \in T - S, \\ 2e & \text{if } i \in S, \\ \epsilon & \text{if } i \in R, \end{cases}$$

$$y_i = \begin{cases} \epsilon & \text{if } i \in S \cup T, \\ (1 - (n - r)e)/r & \text{if } i \in R, \end{cases}$$

$$z_i = \begin{cases} 0 & \text{if } i \in T, \\ 1/(n - t) & \text{if } i \in N - T. \end{cases}$$

Then for sufficiently small $\epsilon > 0$, $x \text{dom}(S) y \text{dom}(T) z$. However, $x$ does not dominate $z$, since $x(T) > v(T)$. Hence $v$ is absolutely stable only if for every minimal-vital $T$, either $v(T) = 1$, or every $S$ with $v(S) > 0$ satisfies either $S \supseteq T$ or $S \supseteq N - T$. This establishes the necessity of (b).

Next, assume that $T$ is minimal-vital with $v(T) < 1$, and assume that (a) fails. Then there is a minimal nonempty coalition $S$ for which $S \cap T = \emptyset$, $S \cup T \not= N$, and $v(S \cup T) > v(T)$. Select any $k \in R = N - (S \cup T)$, and let

$$x_i = \begin{cases} (v(T) + \epsilon)/t & \text{if } i \in T, \\ \epsilon & \text{if } i \in S, \\ 1 - v(T) - (s + 1)\epsilon & \text{if } i = k, \\ 0 & \text{if } i \in R, i \neq k, \end{cases}$$

$$y_i = \begin{cases} \epsilon & \text{if } i \in T, \\ 0 & \text{if } i \in S, \\ (1 - te) & \text{if } i \in R, \end{cases}$$

$$z_i = \begin{cases} 0 & \text{if } i \in T, \\ 1/(n - t) & \text{if } i \in N - T. \end{cases}$$

Then for sufficiently small $\epsilon > 0$, $x \text{dom}(S \cup T) y \text{dom}(T) z$. However, $x \text{dom}(W) z$ for all small $\epsilon$ only if $W = T \cup \{k\}$ and $v(W) = 1$. In this case, let

$$x_i = \begin{cases} (v(T) + \epsilon)/t & \text{if } i \in T, \\ 1 - v(T) - \epsilon & \text{if } i = k, \\ 0 & \text{if } i \in N - W. \end{cases}$$
Then for sufficiently small $\varepsilon > 0$, $x \text{ dom}(W)$ $y \text{ dom}(T)$ $z$, but $x$ does not dominate $z$. Hence $v$ is absolutely stable only if (a) holds. This completes the proof of the theorem.

**Remark.** All simple games (games in which each $v(S)$ is either 0 or 1) satisfy (a) because of monotonicity, and trivially satisfy (b). Games in which all minimal-vital coalitions have at least $n - 1$ players trivially satisfy (a), and are easily seen to satisfy (b). Games of these types were first shown to be absolutely stable in [1]. As an example of a game in neither class, $v$ defined by

\[
\begin{align*}
v({1, 2}) &= 0({1, 2, 3}) = 0({1, 2, 4}) = 1/2, \\
v({1, 3, 4}) &= 1/4, \quad v({2, 3, 4}) = 4/5, \quad v({1, 2, 3, 4}) = 1,
\end{align*}
\]

and $v(S) = 0$ otherwise,

is an absolutely stable four-person game.

**Bibliography**


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