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ABSOLUTELY STABLE GAMES

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ABSTRACT. Absolutely stable games, in which every monotone chain of domination reduces to direct domination, are explicitly characterized. Simple games, and n-person games in which all minimal-vital coalitions contain at least $n - 1$ players, are seen to satisfy the characterization.

Harsanyi [1, pp. 1477–1479] has considered games for which all von Neumann-Morgenstern solutions exhibit a strong form of internal stability. The class of such games includes all absolutely stable games, which Harsanyi defined by a property of chains of domination. It is our purpose to give an explicit characterization of these absolutely stable games.

Let v be a (0, 1)-normalized monotonic game, with player set N. A coalition T is *minimal-vital* if $v(T) > 0$ and if for every $S \subsetneqq T$, $v(S) = 0$. A *monotone chain* is a sequence of imputations and coalitions $(x_0, S_1, x_1, \ldots, S_m, x_m)$ satisfying

(1) x_{k-1} dom(S_k) x_k, and

(2) $(x_0)_i > (x_k)_i$ for all $i \in S_k$,

for $1 \leqslant k \leqslant m$. The game is *absolutely stable* if for every monotone chain $(x_0, S_1, x_1, \ldots, S_m, x_m)$, it follows that x_0 dominates x_m.

THEOREM. *A game v is absolutely stable if and only if*

(a) *for every minimal-vital coalition T, if $N \supsetneqq S \supseteq T$ then $v(S) = v(T)$, and*

(b) *for every minimal-vital coalition T with $v(T) < 1$, every other coalition S with $v(S) > 0$ satisfies either $S \supseteq T$ or $S \supseteq N - T$.*

PROOF. *Sufficiency*. Assume that the conditions are satisfied, and consider a monotone chain $(x_0, S_1, x_1, \ldots, S_m, x_m)$. We will show that x_0 dom(S_m) x_m. By (a), all the coalitions of the chain can be assumed to be minimal-vital. By (2), $(x_0)_i > (x_m)_i$ for all $i \in S_m$. To begin, notice that $x_{m-1}(S_m) \leqslant v(S_m)$, where for notational convenience we generally write $x(S) = \sum_{i \in S}(x)_i$. Proceeding by induction, we assume $x_k(S_m) \leqslant v(S_m)$ and consider x_{k-1}. If $v(S_m) = 1$, then

$$x_{k-1}(S_m) \leqslant x_{k-1}(N) = 1 = v(S_m).$$

Otherwise (b) applies to S_m, and either (i) $S_k \supseteq S_m$, or (ii) $S_k \supseteq N - S_m$.

If (i) applies, then since S_k is minimal-vital, it follows that $S_k = S_m$ and therefore

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$$x_{k-1}(S_m) = x_{k-1}(S_k) \leqslant v(S_k) = v(S_m).$$

On the other hand, if (ii) applies, then $x_{k-1}(N - S_m) > x_k(N - S_m)$, and therefore

$$x_{k-1}(S_m) < x_k(S_m) \leqslant v(S_m),$$

the last inequality following from the induction hypothesis. Thus in any case it eventually follows that $x_0(S_m) \leqslant v(S_m)$, and therefore $x_0 \operatorname{dom}(S_m)\, x_m$.

Necessity. Assume that T is minimal-vital with $v(T) < 1$, and consider any S for which $v(S) > 0$, $S \not\supseteq T$, and $S \not\supseteq N - T$. Then $R = N - (S \cup T) \neq \varnothing$ and $T - S \neq \varnothing$. We write n, r, s, t for the respective cardinalities of N, R, S, T. Let

$$x_i = \begin{cases} (1 - (2s + r)\varepsilon)/(t - s) & \text{if } i \in T - S, \\ 2\varepsilon \quad \text{if } i \in S, \\ \varepsilon \quad \text{if } i \in R, \end{cases}$$

$$y_i = \begin{cases} \varepsilon \quad \text{if } i \in S \cup T, \\ (1 - (n - r)\varepsilon)/r \quad \text{if } i \in R, \end{cases}$$

$$z_i = \begin{cases} 0 \text{ if } i \in T, \\ 1/(n - t) \quad \text{if } i \in N - T. \end{cases}$$

Then for sufficiently small $\varepsilon > 0$, $x \operatorname{dom}(S)\, y \operatorname{dom}(T)\, z$. However, x does not dominate z, since $x(T) > v(T)$. Hence v is absolutely stable only if for every minimal-vital T, either $v(T) = 1$, or every S with $v(S) > 0$ satisfies either $S \supseteq T$ or $S \supseteq N - T$. This establishes the necessity of (b).

Next, assume that T is minimal-vital with $v(T) < 1$, and assume that (a) fails. Then there is a minimal nonempty coalition S for which $S \cap T = \varnothing$, $S \cup T \neq N$, and $v(S \cup T) > v(T)$. Select any $k \in R = N - (S \cup T)$, and let

$$x_i = \begin{cases} (v(T) + \varepsilon)/t \quad \text{if } i \in T, \\ \varepsilon \quad \text{if } i \in S, \\ 1 - v(T) - (s + 1)\varepsilon \quad \text{if } i = k, \\ 0 \quad \text{if } i \in R, i \neq k, \end{cases}$$

$$y_i = \begin{cases} \varepsilon \quad \text{if } i \in T, \\ 0 \quad \text{if } i \in S, \\ (1 - t\varepsilon) \quad \text{if } i \in R, \end{cases}$$

$$z_i = \begin{cases} 0 \quad \text{if } i \in T, \\ 1/(n - t) \quad \text{if } i \in N - T. \end{cases}$$

Then for sufficiently small $\varepsilon > 0$, $x \operatorname{dom}(S \cup T)\, y \operatorname{dom}(T)\, z$. However, $x \operatorname{dom}(W)\, z$ for all small ε only if $W = T \cup \{k\}$ and $v(W) = 1$. In this case, let

$$x_i = \begin{cases} (v(T) + \varepsilon)/t \quad \text{if } i \in T, \\ 1 - v(T) - \varepsilon \quad \text{if } i = k, \\ 0 \quad \text{if } i \in N - W, \end{cases}$$

$$y_i = \begin{cases} \varepsilon & \text{if } i \in T, \\ 0 & \text{if } i = k, \\ (1 - t\varepsilon)/(n - t - 1) & \text{if } i \in N - W, \end{cases}$$

$$z_i = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Then for sufficiently small $\varepsilon > 0$, x dom(W) y dom(T) z, but x does not dominate z. Hence v is absolutely stable only if (a) holds. This completes the proof of the theorem.

REMARK. All simple games (games in which each $v(S)$ is either 0 or 1) satisfy (a) because of monotonicity, and trivially satisfy (b). Games in which all minimal-vital coalitions have at least $n - 1$ players trivially satisfy (a), and are easily seen to satisfy (b). Games of these types were first shown to be absolutely stable in [1]. As an example of a game in neither class, v defined by

$$v(\{1, 2\}) = v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = 1/2,$$
$$v(\{1, 3, 4\}) = 1/4, \quad v(\{2, 3, 4\}) = 4/5, \quad v(\{1, 2, 3, 4\}) = 1,$$

and

$$v(S) = 0 \quad \text{otherwise},$$

is an absolutely stable four-person game.

BIBLIOGRAPHY

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