PROOF OF A CONJECTURE OF FRIEDMAN

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Abstract. We prove that every uncountable hyperarithmetic set has a member of each hyperdegree > 0, the hyperdegree of Kleene's 0.

We improve the main result of Friedman [1] by proving his conjecture that every uncountable hyperarithmetic set has a member of each hyperdegree > 0, the hyperdegree of Kleene's 0. This result has been obtained independently by Friedman by a different method. Friedman's proof uses ideas employed by L. Harrington to obtain a partial result.

In [2] it is shown that there is a function f: \(\omega \to \omega\) of hyperdegree 0 and a Gödel number e such that \((\forall k)(f(k) < g(k))\) implies that f is hyperarithmetic in g with Gödel number e.

By [1] it suffices to prove that each recursive tree of finite sequences of natural numbers with uncountably many branches has a branch of each hyperdegree > 0. Let T be such a tree and let f be as above. Let x: \(\omega \to 2\) have hyperdegree > 0. Let A be the Cantor-Bendixson perfect subtree of T. A E \(\Sigma^1_1\). A point in A is good if it has at least two immediate successors in A.

We shall define a branch h through A. Let \(\sigma_0 < \sigma_1 < \cdots\) be the good points on our branch. ln(\(\sigma\)) is the length of the finite sequence \(\sigma\). \(h(ln(\sigma_n))\) will always be given either the smallest or the next to smallest possible value. \(h(ln(\sigma_{2n}))\) and \(h(ln(\sigma_{2n+1}))\) will have the minimal possible value unless there are exactly k numbers m < n such that

\[
h(ln(\sigma_{2m})) \text{ is not minimal, and } f(k) \leq ln(\sigma_{2n}).
\]

When this happens, \(h(ln(\sigma_{2n}))\) will not be minimal and \(h(ln(\sigma_{2n+1}))\) will be minimal just in case \(x(k) = 0\).

For g: \(\omega \to \omega\) let g E C if and only if, for each k, there are at least \(2k + 1\) good points \(\sigma_n\) of length \(\leq g(k)\) such that \(h(ln(\sigma_n))\) is not minimal. C is \(\Sigma^1_1\) in h and \(g \in C \implies (\forall k)(g(k) \geq f(k))\). f is hyperarithmetic in h since

\[
f^* = f \iff (\exists g \in C) (f^* \text{ is hyperarithmetic in } g \text{ with Gödel number } e).
\]

Since both f and A are hyperarithmetic in each of x and h, x and h are hyperarithmetic in one another.

References


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