PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 1, February 1976

INVERSE LIMITS OF TOPOLOGICAL GROUP COHOMOLOGIES

ARNOLD J. INSEL

ABSTRACT. For second countable locally compact almost connected groups G and A, where A is abelian and G acts on A continuously, it is shown that it is possible to represent A as an inverse limit of Lie groups $\{A_n\}$ compatible with the action of G and such that $H^*(G, A)$ is isomorphic to $\lim_n \text{inv } H^*(G, A_n)$, provided that A is compact or connected.

1. **Introduction**. Let G and A be locally compact second countable groups with A abelian. Suppose G acts on A continuously. In [4] a group cohomology is developed with this situation in mind, $H^*(G, A)$. We assume the definitions and results of [4].

To begin our study we approximate A by an inverse sequence $\{A_n\}$ of Lie groups such that G acts on each A_n in a manner "compatible" with the inverse sequence. We next observe that the inverse sequence $\{A_n\}$ induces an inverse sequence $\{H^*(G, A_n)\}$ of cohomology groups. Considering the "natural" homomorphism of $H^*(G, A)$ to $\lim_n \text{inv}\{H^*(G, A_n)\}$ we investigate the conditions under which the homomorphism is an isomorphism. We find sufficient conditions when A is compact or connected.

2. **Results**. Throughout this paper A and G are locally compact, second countable, Hausdorff, almost connected topological groups and we assume a fixed action of G on A. A group G is almost connected provided that G/G_0 is compact, where G_0 is the connected component of G.

THEOREM 1. *For any neighborhood U of the identity of A there exists a compact subgroup K of A such that*
(1) $K \subset U$,
(2) *K is G-invariant*,
(3) *A/K is a Lie group*.

PROOF. Let $E = A \circledS G$, be the semidirect product of A and G with respect to the action of G on A. Note that topologically E is the product of A and G. We first observe that E/E_0 is almost connected. Let A_0, G_0 and E_0 be the components of A, G and E respectively. It is easily seen that $E_0 = A_0 \times G_0$. Also $E/A_0 \times G_0$ is topologically homeomorphic to $A/A_0 \times G/G_0$–a compact space. Thus E/E_0 is compact. Thus E is almost connected. Let U be any neighborhood of the identity of A. Then $U \times G$ is a neighborhood of the

Presented to the Society, January 17, 1974 under the title *On inverse limits in cohomology groups of topological groups*; received by the editors January 8, 1975.

AMS (MOS) subject classifications (1970). Primary 22D99, 18H10; Secondary 22E99.

Key words and phrases. Inverse limit of Lie groups, topological group cohomologies.

identity on E. Choose a smaller neighborhood $V \times W$ such that $V \times W \subset U \times G$ and $V \times W$ has compact closure in E. Since E is almost connected we conclude by [3] that there exists a closed normal subgroup N of E such that $N \subset V \times W$ and E/N is a Lie group. Let $K' = A \times \{1\} \cap N$ where 1 is the identity of G. Let K be the projection of K' into A. Clearly $K \subset U$, K is compact and G-invariant. By [2, p. 144] we obtain

$$A/K \simeq \frac{A \times \{1\}}{K \times \{1\}} = \frac{A \times \{1\}}{(A \times \{1\}) \cap N} \simeq \frac{(A \times \{1\}) \cdot N}{N}.$$

Since the right-most group is a closed subgroup of the Lie group E/N, A/K is a Lie group.

COROLLARY 1. *There exists a decreasing sequence $\{U_n\}$ of neighborhoods of the identity of A and a decreasing sequence of compact G-invariant subgroups $\{K_n\}$ of A such that*

(1) *$\{U_n\}$ is a decreasing base for the neighborhood system of the identity,*
(2) *$K_n \subset U_n$ for all n,*
(3) *A/K_n is a Lie group for all n.*

PROOF. The above corollary follows immediately from Theorem 1. The only difficulty is to make $\{K_n\}$ into a decreasing sequence. But this is immediate once we observe by [3] that if A/J and A/K are Lie groups so is $A/J \cap K$. Use Theorem 1 to obtain a sequence J_n satisfying (2) and (3) (but not necessarily decreasing). Then define $K_n = \bigcap_{i=1}^{n} J_i$ for all n.

COROLLARY 2. *There exists a sequence of abelian Lie groups $\{A_n\}$ and continuous surjective homomorphisms $\{\pi_n\}$ and $\{\rho_n\}$ such that:*

(1) *G acts on A_n for all n.*
(2) *$\pi_n\colon A_n \to A_{n-1}$ and $\pi_n(ga) = g\pi_n(a)$ for all $a \in A_n$ and $g \in G$.*
(3) *$\rho_n\colon A \to A_n$ and $\rho_n(ga) = g\rho_n(a)$ for all $a \in A_n$ and $g \in G$.*
(4) *$\pi_n\rho_n = \pi_{n-1}$ for all n.*
(5) *Define $\phi\colon A \to \lim_n \mathrm{inv}\, A_n$ by $\phi(a) = \{\rho_n(a)\}$. Then ϕ is a topological isomorphism and $\phi(ga) = g\phi(a)$ for all $a \in A$ and $g \in G$.*

PROOF. Referring to Corollary 1 define $A_n = A/K_n$ for all n. For each n let $\rho_n\colon A \to A_n$ be the natural quotient map and for each n let $\pi_n\colon A_n \to A_{n-1}$ be the map $\pi_n(x \cdot K_n) = x \cdot K_{n-1}$, $x \in A$. Clearly the corollary follows.

For the rest of this paper we assume the inverse system of Lie groups as constructed in Corollary 2.

On the cohomology level we inherit an inverse sequence $\{H^*(G\ A_n)\}$ with homomorphisms $\{\pi_{n_*}\}$ and $\{\rho_{n_*}\}$ induced by the homomorphisms $\{\pi_n\}$ and $\{\rho_n\}$ such that $\pi_{n_*}\rho_{n_*} = \pi_{n-1_*}$ for all n. For each k define

$$\Phi_A\colon H^k(G, A) \to \lim_n \mathrm{inv}\big\{H^k(G, A_n)\big\}$$

by

$$\Phi_A(h) = \big\{\rho_{n_*}(h)\big\}, \qquad h \in H^k(G, A).$$

The next theorem will enable us to conclude that Φ_A is onto. To facilitate its proof let us recall some notation. For each k, $Z^k(G, A)$, $B^k(G, A)$ and

$C^k(G, A)$ will denote the groups of k-cocycles, k-coboundaries and k-cochains of G over A respectively. Also δ_k: $C^k(G, A) \to C^{k+1}(G, A)$ will denote the kth coboundary homomorphism. For definitions see [4]. For $\sigma \in Z^k(G, A)$, $[\sigma]$ will denote the element of $H^k(G, A)$ to which σ belongs. If $h \in H^k(G, A)$ and if $\sigma_1, \sigma_2 \in Z^n(G, A)$ such that $[\sigma_1] = [\sigma_2] = h$ we say that σ_1 and σ_2 are cohomologous. Recall also that a sequence $\{h_n\}$ lies in $\lim_n \mathrm{inv}\{H^k(G, A_n)\}$ if $h_n \in H^k(G, A_n)$ for all n and if $\pi_{n_*}(h_n) = h_{n-1}$ for all n. We shall use the same symbols π_{n_*} and ρ_{n_*} to denote the homomorphisms induced on the cohomology level and on the cochain level from π_n and ρ_n. We now state the theorem.

THEOREM 2. *For any integer $k \geq 0$ let $\{h_n\} \in \lim_n \mathrm{inv}\{H^k(G, A_n)\}$. Then for each n there exists a $\sigma_n \in Z^k(G, A_n)$ such that*
(1) $\pi_{n_*}(\sigma_n) = \sigma_{n-1}$,
(2) $[\sigma_n] = h_n$.

PROOF. We use induction on n. Let σ_1 be any member of h_1. Now suppose we have succeeded in choosing a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ satisfying conditions (1) and (2) for $1 \leq j \leq n$. Choose $\sigma'_{n+1} \in h_{n+1}$. Since $\pi_{n+1_*}(\sigma'_{n+1})$ is cohomologous to σ_n there exists a Borel function $\zeta_n \in C^{k-1}(G, A_n)$ such that $\pi_{n+1_*}(\sigma'_{n+1}) + \delta_{k-1}(\zeta) = \sigma_n$. Let $\lambda_n : A_n \longrightarrow A_{n+1}$ be a Borel cross section of π_{n+1} such that $\lambda_n(1) = 1$ (see [1]). Define ψ_{n+1} to be the composition $\psi_{n+1} = \lambda_n \zeta_n$. Note that $\psi_{n+1} \in C^{k-1}(G, A_{n+1})$. Define $\sigma_{n+1} = \sigma'_{n+1} + \delta_{k-1}(\psi_{n+1})$. It immediately follows that $[\sigma_{n+1}] = h_{n+1}$ and

$$\begin{aligned} \pi_{n+1_*}(\sigma_{n+1}) &= \pi_{n+1_*}[\sigma'_{n+1} + \delta_{k-1}(\psi_{n+1})] \\ &= \pi_{n+1_*}(\sigma'_{n+1}) + \delta_{k-1}(\zeta_n) = \sigma_n. \end{aligned}$$

Continuing in this manner we construct the desired sequence.

COROLLARY. *The homomorphism Φ_A: $H^*(G, A) \to \lim_n \mathrm{inv}\{H^*(G, A_n)\}$ is onto.*

PROOF. Let $\{h_n\} \in \lim_n \mathrm{inv}\{H^k(G, A_n)\}$ for some k. Let $\{\sigma_n\}$ be a sequence satisfying the conditions of Theorem 2. By condition (1) of Theorem 2 for any $(g_1, g_2, \ldots, g_k) \in G^k$, $\{\sigma_n(g_1, g_2, \ldots, g_k)\} \in \lim_n \mathrm{inv}\{A_n\}$. Thus $\phi^{-1}(\{\sigma_n(g_1, \ldots, g_k)\}) \in A$ where ϕ is the topological isomorphism of Corollary 2 to Theorem 1. Now define $\sigma \in C^k(G, A)$ by $\sigma(g_1, \ldots, g_k) = \phi^{-1}(\{\sigma_n(g_1, \ldots, g_k)\})$ for $(g_1, \ldots, g_k) \in G^k$. It easily follows that $\sigma \in Z^k(G, A)$ and if $h = [\sigma]$, $\Phi_A(h) = \{h_n\}$. Thus Φ_A is onto.

We now turn to sufficient conditions for Φ_A to be one-to-one. First note that $B^k(G, A)$, $Z^k(G, A)$ and $C^k(G, A)$ are all subsets of A^{G^k}, the set of all mappings of G^k to A. As such they inherit topologies as subspaces of A^{G^k} with the product topology.

THEOREM 3. *If $B^k(G, A)$ is a closed subset of the topological space $C^k(G, A)$ for all k then Φ_A is one-to-one and hence is an isomorphism.*

PROOF. Suppose the condition holds. Let $k \geq 0$. Let $h \in H^k(G, A)$ such that $\Phi_A(h) = 0$. We show $h = 0$.

By the exactness of the sequence

$$1 \to K_n \xrightarrow{i_n} A \xrightarrow{\rho_n} A_n \to 1$$

and by [4] we obtain the exactness of

$$\to H^k(G, K_n) \xrightarrow{i_{n_*}} H^k(G, A) \xrightarrow{\rho_{n_*}} H^k(G, A_n) \to.$$

Since $\rho_{n_*}(h) = 0$ for all n there exists for each n (by exactness) a $c_n \in H^k(G, K_n)$ such that $i_{n_*}(c_n) = h$. Choose cocycles $\sigma \in h$ and $\beta_n \in c_n$ for each n. Since $i_{n_*}(\beta_n)$ is cohomologous to h for each n, $i_{n_*}(\beta_n) \in \sigma + B^k(G, A)$–a closed subset of $C^k(G, A)$. For each n, β_n maps G^k into K_n. Since $K_n \subset U_n$ and $\{U_n\}$ is a decreasing base for the neighborhood system of the identity of A we have

$$\lim_{n\to\infty} i_{n_*}\beta_n(g_1, \ldots, g_k) = \lim_{n\to\infty} i_n \circ \beta_n(g_1, \ldots, g_k) = 1$$

for each $(g_1, g_2, \ldots, g_k) \in G^k$. Thus the sequence $\{i_{n_*}\beta_n\}$ converges to the trivial mapping, 0, in the product topology. Thus by closure, $0 \in \sigma + B^k(G, A)$. Thus $[\sigma] = 0$. Thus Φ_A is one-to-one.

THEOREM 4. *If A is compact, $B^k(G, A)$ is closed in $C^k(G, A)$ and hence Φ_A is an isomorphism.*

PROOF. First define for each k:
(1) $C_N^k(G, A) = \{f: G^k \to A \mid \text{if } g_i = 1 \text{ for some } i, f(g_1, \ldots, g_k) = 1\}$.
(2) δ_N^k: $C_N^k(G, A) \to C_N^{k+1}(G, A)$ by

$$\begin{aligned}\delta_k^N(g_1, \ldots, g_{k+1}) = {} & g_1 f(g_2, \ldots, g_{k+1}) \\ & + \sum_{i=1}^{k} (-1)^i f(g_1, \ldots, g_i \cdot g_{i+1}, \ldots, g_{k+1}) \\ & + (-1)^{k+1} f(g_1, \ldots, g_k).\end{aligned}$$

(3) $Z_N^k(G, A)$–the kernel of δ_k^N.
(4) $B_N^k(G, A)$–the image of δ_{k-1}^N.
It is easily verified that $C_N^k(G, A)$ is closed in A^{G^k} and hence is compact since A is compact. It is also easily verified that δ_k^N is continuous for each k. Thus $B_N^k(G, A)$ and $Z_N^k(G, A)$ are both closed subsets of $C_N^k(G, A)$. Let B denote the set of all Borel functions in A^{G^k}. Then
(5) $C^k(G, A) = C_N^k(G, A) \cap B$.
(6) $\delta_k = \delta_k^N$ restricted to $C^k(G, A)$.
(7) $Z^k(G, A) = Z_N^k(G, A) \cap B$.
(8) $B^k(G, A) = B_N^k(G, A) \cap B$.
Thus $B^k(G, A)$ is closed in $C^k(G, A)$. The result now follows from Theorem 3.

THEOREM 5. *If A is connected Φ_A is an isomorphism.*

PROOF. Suppose A is connected. By [5] A splits as a direct sum of a compact group C and a finite product R^p of copies of the reals. Thus we have a splitting sequence $1 \to C \xrightarrow{i} A \xrightarrow{j} R^p \to 1$.

Note that $i(C)$ is invariant with respect to the action of G since R^p contains no nontrivial compact group. Thus G induces an action on C

($g \cdot c = i^{-1}(g \cdot i(c))$, $c \in C$, $g \in G$) and an action on R^p ($g \cdot r = j(g \cdot j^{-1}(r))$, $r \in R^p$ and $g \in G$). These actions are compatible with the homomorphisms i and j.

Consider the collections $\{K_n\}$ and $\{U_n\}$ of Corollary 1 to Theorem 1. Note that $K_n \subset i(C)$ for each n since K_n is compact. For each n define $H_n = i^{-1}(K_n)$. Note that C/H_n is a Lie group. This is so since $C/H_n \simeq i(C)/K_n$–a closed subgroup of A/K_n.

For each n let $V_n = i^{-1}(i(C) \cap U_n)$. Then $\{V_n\}$ and $\{H_n\}$ are decreasing sequences and

(1) $\{V_n\}$ is a decreasing base for the neighborhood system of the identity of C.

(2) $H_n \subset V_n$ for all n.

(3) H_n is a G-invariant compact subgroup of C for each n.

(4) C/H_n is a Lie group for each n.

Thus Corollary 2 of Theorem 1 follows for Lie groups $\{C_n\}$ and sequences of homomorphisms $\{\rho_n\}$ and $\{\pi_n\}$ where $C_n = C/H_n$, ρ_n: $C \to C_n$ is the natural quotient map and $\pi_n(x \cdot C_n) = x \cdot C_{n-1}$ for $x \in C$, for all n. Furthermore we obtain for each n the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \to & C_n & \xrightarrow{i_n} & A_n & \xrightarrow{j_n} & R^p \\ \uparrow & & \uparrow\rho_n & & \uparrow\rho_n & & \uparrow\rho \\ 1 & \to & C & \xrightarrow{i} & A & \xrightarrow{j} & R^p \end{array}$$

where $i_n(x \cdot H_n) = i(x)K_n$ for $x \cdot H_n \in C_n$, j_n is the quotient map and ρ is the identity.

By [4] we obtain the following commutative diagram of cohomology groups:

$$\begin{array}{ccccc} \to H^k(G, C_n) & \xrightarrow{i_{n*}} & H^k(G, A_n) & \xrightarrow{j_{n*}} & H^k(G, R^p) \\ \uparrow\rho_{n*} & & \uparrow\rho_{n*} & & \uparrow\rho_* \\ \to H^k(G, C) & \xrightarrow{i_*} & H^k(G, A) & \xrightarrow{j_*} & H^k(G, R^p) \\ \uparrow q_{n*} & & \uparrow q_{n*} & & \\ H^k(G, H_n) & \xrightarrow{i_*} & H^k(G, K_n) & & \end{array}$$

where q_n denote both the injections of H_n into C and of K_n into A. Note that since i: $H_n \to K_n$ is an isomorphism so is i_*: $H^k(G, H_n) \to H^k(G, K_n)$.

We wish to show that Φ_A is one-to-one. Let $h \in H^k(G, A)$. Suppose $\Phi_A(h) = 0$. Then $\rho_{n_*}(h) = 0$ for all n. It suffices to show that $h = 0$.

First note that $\rho_{1_*}(h) = 0$. Hence $\rho_* j_*(h) = j_{1_*}\rho_{1_*}(h) = 0$. Since ρ_* is an isomorphism, $j_*(h) = 0$. Thus by exactness there exists a $c \in H^k(G, C)$ such that $i_*(c) = h$. Choose $\gamma \in Z^n(G, C)$ such that $\gamma \in c$. Let $\sigma = i_*(\gamma)$. Then $\sigma \in h$ and $\sigma \in i_*[Z^k(G, C)]$. For each n, since $\rho_{n_*}(h) = 0$ we can choose $c_n \in H^k(G, K_n)$ such that $q_{n_*}(c_n) = h$. By commutativity we can choose $\beta_n \in Z^k(G, H_n)$ such that $i_* q_{n*}(\beta_n) \in h$. Let $x_n = q_{n_*}(\beta_n)$. Then $i_*(\beta_n) \in$

$\sigma + B^k(G, A)$. Since both σ and $i_*(x_n) \in i_*[Z^k(B, C)]$, $i_*(x_n) \in \sigma + i_*[B^k(B, C)]$.

Since $\bar{\beta}_n$: $G^k \to H_n \subset V_n$ and since V_n is a decreasing base for the neighborhood system of the identity in C we have that $\{\beta_n\} \to 0$ pointwise. Hence $\{x_n\} = \{q_{n_*}(\beta_n)\} \to 0$ pointwise. Hence $\{i_*(x_n)\} \to 0$ pointwise. Next observe that $i_*[B^k(G, C)]$ is closed in $C^k(G, A)$ in the product topology. This follows (using the same terminology as in the proof of Theorem 4) since $B_N^k(G, C)$ is a compact subset of $C_N^k(G, C)$ and i_* is continuous. Thus $i_*[B_N^k(G, C)]$ is compact and hence closed in $C_N^k(G, A)$. Finally

$$i_*\left[B^k(G, C)\right] = i_*\left[B_N^k(G, C)\right] \cap B$$

is closed in $C^k(G, A)$. Thus $\sigma + i_*[B^k(G, C)]$ is closed and the sequence $\{i_*(x_n)\}$ in this set converges to 0. Thus

$$0 \in \sigma + i_*[B^k(G, C)] \subset \sigma + B^k(G, A).$$

Thus $h = 0$. This establishes the theorem.

References

1. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc. no. 62(1966). MR**34** #7723.

2. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR**28** #158.

3. D. Montgomery and L. Zippen, *Topological transformation groups*, Interscience, New York, 1955. MR**17**, 383.

4. C. C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups*. I, Trans. Amer. Math. Soc. **113**(1964), 40–63. MR**30** #2106.

5. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR**27** #2808.

Department of Mathematics, Illinois State University, Normal, Illinois 61761