INVERSE LIMITS OF TOPOLOGICAL GROUP COHOMOLOGIES

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Abstract. For second countable locally compact almost connected groups \( G \) and \( A \), where \( A \) is abelian and \( G \) acts on \( A \) continuously, it is shown that it is possible to represent \( A \) as an inverse limit of Lie groups \( \{ A_n \} \) compatible with the action of \( G \) and such that \( H^*(G, A) \) is isomorphic to \( \lim_n \text{inv} \ H^*(G, A_n) \), provided that \( A \) is compact or connected.

1. Introduction. Let \( G \) and \( A \) be locally compact second countable groups with \( A \) abelian. Suppose \( G \) acts on \( A \) continuously. In [4] a group cohomology is developed with this situation in mind, \( H^*(G, A) \). We assume the definitions and results of [4].

To begin our study we approximate \( A \) by an inverse sequence \( \{ A_n \} \) of Lie groups such that \( G \) acts on each \( A_n \) in a manner "compatible" with the inverse sequence. We next observe that the inverse sequence \( \{ A_n \} \) induces an inverse sequence \( \{ H^*(G, A_n) \} \) of cohomology groups. Considering the "natural" homomorphism of \( H^*(G, A) \) to \( \lim_n \text{inv} \{ H^*(G, A_n) \} \) we investigate the conditions under which the homomorphism is an isomorphism. We find sufficient conditions when \( A \) is compact or connected.

2. Results. Throughout this paper \( A \) and \( G \) are locally compact, second countable, Hausdorff, almost connected topological groups and we assume a fixed action of \( G \) on \( A \). A group \( G \) is almost connected provided that \( G/G_0 \) is compact, where \( G_0 \) is the connected component of \( G \).

Theorem 1. For any neighborhood \( U \) of the identity of \( A \) there exists a compact subgroup \( K \) of \( A \) such that

1. \( K \subset U \),
2. \( K \) is \( G \)-invariant,
3. \( A/K \) is a Lie group.

Proof. Let \( E = A \circledast G \), be the semidirect product of \( A \) and \( G \) with respect to the action of \( G \) on \( A \). Note that topologically \( E \) is the product of \( A \) and \( G \). We first observe that \( E/E_0 \) is almost connected. Let \( A_0, G_0 \) and \( E_0 \) be the components of \( A, G \) and \( E \) respectively. It is easily seen that \( E_0 = A_0 \times G_0 \). Also \( E/A_0 \times G_0 \) is topologically homeomorphic to \( A/A_0 \times G/G_0 \) a compact space. Thus \( E/E_0 \) is compact. Thus \( E \) is almost connected. Let \( U \) be any neighborhood of the identity of \( A \). Then \( U \times G \) is a neighborhood of the

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identity on $E$. Choose a smaller neighborhood $V \times W$ such that $V \times W \subset U \times G$ and $V \times W$ has compact closure in $E$. Since $E$ is almost connected we conclude by [3] that there exists a closed normal subgroup $N$ of $E$ such that $N \subset V \times W$ and $E/N$ is a Lie group. Let $K' = A \times \{1\} \cap N$ where 1 is the identity of $G$. Let $K$ be the projection of $K'$ into $A$. Clearly $K \subset U$, $K$ is compact and $G$-invariant. By [2, p. 144] we obtain

$$ \frac{A \times \{1\}}{K \times \{1\}} = \frac{A \times \{1\}}{(A \times \{1\}) \cap N} \simeq \frac{(A \times \{1\}) \cdot N}{N}. $$

Since the right-most group is a closed subgroup of the Lie group $E/N$, $A/K$ is a Lie group.

**Corollary 1.** There exists a decreasing sequence $\{U_n\}$ of neighborhoods of the identity of $A$ and a decreasing sequence of compact $G$-invariant subgroups $\{K_n\}$ of $A$ such that

1. $\{U_n\}$ is a decreasing base for the neighborhood system of the identity,
2. $K_n \subset U_n$ for all $n$,
3. $A/K_n$ is a Lie group for all $n$.

**Proof.** The above corollary follows immediately from Theorem 1. The only difficulty is to make $\{K_n\}$ into a decreasing sequence. But this is immediate once we observe by [3] that if $A/J$ and $A/K$ are Lie groups so is $A/J \cap K$. Use Theorem 1 to obtain a sequence $J_n$ satisfying (2) and (3) (but not necessarily decreasing). Then define $K_n = \bigcap_{i=1}^n J_i$ for all $n$.

**Corollary 2.** There exists a sequence of abelian Lie groups $\{A_n\}$ and continuous surjective homomorphisms $\{\pi_n\}$ and $\{\rho_n\}$ such that:

1. $G$ acts on $A_n$ for all $n$.
2. $\pi_n: A_n \to A_{n-1}$ and $\pi_n(ga) = g\pi_n(a)$ for all $a \in A_n$ and $g \in G$.
3. $\rho_n: A \to A_n$ and $\rho_n(ga) = g\rho_n(a)$ for all $a \in A_n$ and $g \in G$.
4. $\pi_n \rho_n = \pi_{n-1}$ for all $n$.
5. Define $\phi: A \to \lim_n \inv A_n$ by $\phi(a) = \{\rho_n(a)\}$. Then $\phi$ is a topological isomorphism and $\phi(ga) = g\phi(a)$ for all $a \in A$ and $g \in G$.

**Proof.** Referring to Corollary 1 define $A_n = A/K_n$ for all $n$. For each $n$ let $\rho_n: A \to A_n$ be the natural quotient map and for each $n$ let $\pi_n: A_n \to A_{n-1}$ be the map $\pi_n(x \cdot K_n) = x \cdot K_{n-1}$, $x \in A$. Clearly the corollary follows.

For the rest of this paper we assume the inverse system of Lie groups as constructed in Corollary 2.

On the cohomology level we inherit an inverse sequence $\{H^*(G, A_n)\}$ with homomorphisms $\{\pi_{n*}\}$ and $\{\rho_{n*}\}$ induced by the homomorphisms $\{\pi_n\}$ and $\{\rho_n\}$ such that $\pi_{n*} \rho_{n*} = \pi_{n-1*}$ for all $n$. For each $k$ define $\Phi_A: H^k(G, A) \to \lim_n \inv \{H^k(G, A_n)\}$ by

$$ \Phi_A(h) = \{\rho_{n*}(h)\}, \quad h \in H^k(G, A). $$

The next theorem will enable us to conclude that $\Phi_A$ is onto. To facilitate its proof let us recall some notation. For each $k$, $Z^k(G, A)$, $B^k(G, A)$ and
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C^k(G, A) will denote the groups of k-cocycles, k-coboundaries and k-cochains of G over A respectively. Also δ_k: C^k(G, A) → C^{k+1}(G, A) will denote the kth coboundary homomorphism. For definitions see [4]. For σ ∈ Z^k(G, A), [σ] will denote the element of H^k(G, A) to which σ belongs. If h ∈ H^k(G, A) and if σ_1, σ_2 ∈ Z^n(G, A) such that [σ_1] = [σ_2] = h we say that σ_1 and σ_2 are cohomologous. Recall also that a sequence \{h_n\} lies in \lim_{n \to \infty} \{H^k(G, A_n)\} if h_n ∈ H^k(G, A_n) for all n and if π_n^*(h_n) = h_{n+1} for all n. We shall use the same symbols π_n and ρ_n to denote the homomorphisms induced on the cohomology level and on the cochain level from π_n and ρ_n.

We now state the theorem.

**Theorem 2.** For any integer k > 0 let \{h_n\} ∈ lim_n inv{H^k(G, A_n)}. Then for each n there exists a o_n ∈ Z^k(G, A_n) such that

1. π_n(o_n) = σ_n-1,
2. [σ_n] = h_n.

**Proof.** We use induction on n. Let σ_1 be any member of h_1. Now suppose we have succeeded in choosing a sequence σ_1, σ_2, . . . , σ_n satisfying conditions (1) and (2) for 1 ≤ j ≤ n. Choose σ'_{n+1} ∈ h_{n+1}. Since π_{n+1}^*(σ'_{n+1}) is cohomologous to σ_n there exists a Borel function ζ_n ∈ C^{k-1}(G, A_n) such that π_{n+1}^*(σ'_{n+1}) + δ_{k-1}(ζ_n) = σ_n. Let λ_n: A → A_{n+1} be a Borel cross section of π_{n+1} such that λ_n(1) = 1 (see [1]). Define ψ_{n+1} to be the composition ψ_{n+1} = λ_nζ_n. Note that ψ_{n+1} ∈ C^{k-1}(G, A_{n+1}). Define σ_{n+1} = o'_{n+1} + δ_{k-1}(ψ_{n+1}). It immediately follows that [σ_{n+1}] = h_{n+1} and

π_{n+1}^*(σ_{n+1}) = π_{n+1}^*[σ'_{n+1} + δ_{k-1}(ψ_{n+1})]

= π_{n+1}^*(σ'_{n+1}) + δ_{k-1}(ζ_n) = σ_n.

Continuing in this manner we construct the desired sequence.

**Corollary.** The homomorphism Φ_A: H^*(G, A) → lim_n inv{H^*(G, A_n)} is onto.

**Proof.** Let \{h_n\} ∈ lim_{n \to \infty} \{H^k(G, A_n)\} for some k. Let \{σ_n\} be a sequence satisfying the conditions of Theorem 2. By condition (1) of Theorem 2 for any (g_1, g_2, . . . , g_k) ∈ G^k, {σ_n(g_1, g_2, . . . , g_k)} ∈ lim_{n \to \infty} \{A_n\}. Thus φ^{-1}((σ_n(g_1, g_2, . . . , g_k))) ∈ A where φ is the topological isomorphism of Corollary 2 to Theorem 1. Now define σ ∈ C^k(G, A) by σ(g_1, . . . , g_k) = φ^{-1}((σ_n(g_1, g_2, . . . , g_k))) for (g_1, . . . , g_k) ∈ G^k. It easily follows that σ ∈ Z^k(G, A) and if h = [σ], Φ_A(h) = {h_n}. Thus Φ_A is onto.

We now turn to sufficient conditions for Φ_A to be one-to-one. First note that B^k(G, A), Z^k(G, A) and C^k(G, A) are all subsets of A^G^k, the set of all mappings of G^k to A. As such they inherit topologies as subspaces of A^G^k with the product topology.

**Theorem 3.** If B^k(G, A) is a closed subset of the topological space C^k(G, A) for all k then Φ_A is one-to-one and hence is an isomorphism.

**Proof.** Suppose the condition holds. Let k > 0. Let h ∈ H^k(G, A) such that Φ_A(h) = 0. We show h = 0.

By the exactness of the sequence.
and by [4] we obtain the exactness of

$$i_n \cdot \rho_n : H^k(G, K_n) \to H^k(G, A) \to H^k(G, A_n) \to .$$

Since $\rho_n(h) = 0$ for all $n$ there exists for each $n$ (by exactness) a $c_n \in H^k(G, K_n)$ such that $i_n(c_n) = h$. Choose cocycles $\sigma \in h$ and $\beta_n \in c_n$ for each $n$. Since $i_n(\beta_n)$ is cohomologous to $h$ for each $n$, $i_n(\beta_n) \in \sigma + B^k(G, A)$—a closed subset of $C^k(G, A)$. For each $n$, $\beta_n$ maps $G^n$ into $K_n$. Since $K_n \subset U_n$ and $\{ U_n \}$ is a decreasing base for the neighborhood system of the identity of $A$ we have

$$\lim_{n \to \infty} i_n \beta_n(g_1, \ldots, g_k) = \lim_{n \to \infty} i_n \circ \beta_n(g_1, \ldots, g_k) = 1$$

for each $(g_1, g_2, \ldots, g_k) \in G^k$. Thus the sequence $\{ i_n \beta_n \}$ converges to the trivial mapping, 0, in the product topology. Thus by closure, $0 \in \sigma + B^k(G, A)$. Thus $[\sigma] = 0$. Thus $\Phi_A$ is one-to-one.

**Theorem 4.** If $A$ is compact, $B^k(G, A)$ is closed in $C^k(G, A)$ and hence $\Phi_A$ is an isomorphism.

**Proof.** First define for each $k$:

1. $C^k(G, A) = \{ f : G \times G^k \to A \mid f(g, 1, \ldots, g_k) = 1 \}$.  
2. $\delta^N_k : C^k(G, A) \to C^{k+1}(G, A)$ by
   
   $$\delta^N_k(g_1, \ldots, g_{k+1}) = g_1 f(g_2, \ldots, g_{k+1})$$
   
   $$+ \sum_{i=1}^{k} (-1)^i f(g_1, \ldots, g_i \cdot g_{i+1}, \ldots, g_{k+1})$$
   
   $$+ (-1)^{k+1} f(g_1, \ldots, g_k).$$

3. $Z^k(G, A)$—the kernel of $\delta^N_k$.
4. $B^k(G, A)$—the image of $\delta^{k-1}_N$.

It is easily verified that $C^k(G, A)$ is closed in $A G^k$ and hence is compact since $A$ is compact. It is also easily verified that $\delta^N_k$ is continuous for each $k$. Thus $B^k(G, A)$ and $Z^k(G, A)$ are both closed subsets of $C^k(G, A)$. Let $B$ denote the set of all Borel functions in $A G^k$. Then

5. $C^k(G, A) = C^k(G, A) \cap B$.
6. $\delta_k = \delta^N_k$ restricted to $C^k(G, A)$.
7. $Z^k(G, A) = Z^k(G, A) \cap B$.
8. $B^k(G, A) = B^k(G, A) \cap B$.

Thus $B^k(G, A)$ is closed in $C^k(G, A)$. The result now follows from Theorem 3.

**Theorem 5.** If $A$ is connected $\Phi_A$ is an isomorphism.

**Proof.** Suppose $A$ is connected. By [5] $A$ splits as a direct sum of a compact group $C$ and a finite product $R^p$ of copies of the reals. Thus we have a splitting sequence $1 \to C \to A \to R^p \to 1$.

Note that $i(C)$ is invariant with respect to the action of $G$ since $R^p$ contains no nontrivial compact group. Thus $G$ induces an action on $C$
(g \cdot c = i^{-1}(g \cdot i(c)), c \in C, g \in G) and an action on \( R^p \) \((g \cdot r = j(g \cdot j^{-1}(r)), r \in R^p and g \in G)\). These actions are compatible with the homomorphisms \( i \) and \( j \).

Consider the collections \( \{K_n\} \) and \( \{U_n\} \) of Corollary 1 to Theorem 1. Note that \( K_n \subset i(C) \) for each \( n \) since \( K_n \) is compact. For each \( n \) define \( H_n = i^{-1}(K_n) \). Note that \( C/H_n \) is a Lie group. This is so since \( C/H_n \simeq i(C)/K_n \) a closed subgroup of \( A/K_n \).

For each \( n \) let \( V_n = i^{-1}(i(C) \cap U_n) \). Then \( \{V_n\} \) and \( \{H_n\} \) are decreasing sequences and

1. \( \{V_n\} \) is a decreasing base for the neighborhood system of the identity of \( C \).
2. \( H_n \subset V_n \) for all \( n \).
3. \( H_n \) is a \( G \)-invariant compact subgroup of \( C \) for each \( n \).
4. \( C/H_n \) is a Lie group for each \( n \).

Thus Corollary 2 of Theorem 1 follows for Lie groups \( \{C_n\} \) and sequences of homomorphisms \( \{\rho_n\} \) and \( \{\pi_n\} \) where \( C_n = C/H_n, \rho_n: C \rightarrow C_n \) is the natural quotient map and \( \pi_n(x \cdot C_n) = x \cdot C_{n-1} \) for \( x \in C \), for all \( n \). Furthermore we obtain for each \( n \) the following commutative diagram:

\[
\begin{array}{ccc}
1 & \rightarrow & C_n \\
\uparrow & & \uparrow \rho_n \\
1 & \rightarrow & C
\end{array}
\quad \begin{array}{ccc}
i_n & \rightarrow & A_n \\
\uparrow & & \uparrow \rho_n \\
( & \rightarrow & A
\end{array}
\quad \begin{array}{ccc}
j_n & \rightarrow & R^p \\
\uparrow & & \uparrow \rho \\
( & \rightarrow & R^p
\end{array}
\]

where \( i_n(x \cdot H_n) = i(x)K_n \) for \( x \cdot H_n \in C_n \), \( j_n \) is the quotient map and \( \rho \) is the identity.

By [4] we obtain the following commutative diagram of cohomology groups:

\[
\begin{array}{ccc}
H^k(G, C_n) & \rightarrow & H^k(G, A_n) \\
\uparrow q_n \ast & & \uparrow q_n \ast \\
H^k(G, C) & \rightarrow & H^k(G, A)
\end{array}
\quad \begin{array}{ccc}
i_n \ast & \rightarrow & j_n \ast \\
\uparrow \rho_n \ast & & \uparrow \rho \ast \\
( & \rightarrow & R^p
\end{array}
\]

where \( q_n \) denote both the injections of \( H_n \) into \( C \) and of \( K_n \) into \( A \). Note that since \( i: H_n \rightarrow K_n \) is an isomorphism so is \( i_n: H^k(G, H_n) \rightarrow H^k(G, K_n) \).

We wish to show that \( \Phi_A \) is one-to-one. Let \( h \in H^k(G, A) \). Suppose \( \Phi_A(h) = 0 \). Then \( \rho_n \ast(h) = 0 \) for all \( n \). It suffices to show that \( h = 0 \).

First note that \( \rho_1 \ast(h) = 0 \). Hence \( \rho_2 \ast j_2 \ast(h) = j_1 \ast \rho_1 \ast(h) = 0 \). Since \( \rho_2 \ast \) is an isomorphism, \( j_1 \ast(h) = 0 \). Thus by exactness there exists a \( c \in H^k(G, C) \) such that \( i_1 \ast(c) = h \). Choose \( \gamma \in Z^n(G, C) \) such that \( \gamma \in c \). Let \( \gamma = i_1 \ast(\gamma) \). Then \( \sigma \in h \) and \( \sigma \in i_1 \ast[Z^k(G, C)] \). For each \( n \), since \( \rho_n \ast(h) = 0 \) we can choose \( c_n \in H^k(G, K_n) \) such that \( q_n \ast(c_n) = h \). By commutativity we can choose \( \beta_n \in Z^k(G, H_n) \) such that \( i_n \ast q_n \ast(\beta_n) \in h \). Let \( x_n = q_n \ast(\beta_n) \). Then \( i_n \ast(\beta_n) \in H^k(G, C_n) \)
σ + B^k(G, A). Since both σ and i_*(x_n) ∈ i_*[Z^k(B, C)], i_*(x_n) ∈ σ + i_*[B^k(B, C)].

Since β_n: G^k → H_n ⊂ V_n and since V_n is a decreasing base for the neighborhood system of the identity in C we have that {β_n} → 0 pointwise. Hence {x_n} = {q_n(β_n)} → 0 pointwise. Hence {i_*(x_n)} → 0 pointwise. Next observe that i_*[B^k(G, C)] is closed in C^k(G, A) in the product topology. This follows (using the same terminology as in the proof of Theorem 4) since B^k_N(G, C) is a compact subset of C^k_N(G, C) and i_* is continuous. Thus i_*[B^k_N(G, C)] is compact and hence closed in C^k_N(G, A). Finally

i_*[B^k(G, C)] = i_*[B^k_N(G, C)] ∩ B

is closed in C^k(G, A). Thus σ + i_*[B^k(G, C)] is closed and the sequence {i_*(x_n)} in this set converges to 0. Thus

0 ∈ σ + i_*[B^k(G, C)] ⊂ σ + B^k(G, A).

Thus h = 0. This establishes the theorem.

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