UNDECIDABILITY OF THE THEORY OF
ABELIAN GROUPS WITH A SUBGROUP

WALTER BAUR

Abstract. The theory of abelian groups with an additional predicate denoting a subgroup is undecidable.

0. Introduction. Let \( L \) be the first order language with nonlogical symbols \( 0, +, P \), where \( P \) is a unary predicate symbol. For any class \( K \) of abelian groups let \( T(K) \) denote the \( L \)-theory of the class of structures \( \langle A, B \rangle \), where \( A \in K \) and \( B \subseteq A \) is an arbitrary subgroup. Kozlov and Kokorin [4] showed that \( T(K) \) is decidable if \( K \) is the class of torsion free groups. The main result of this paper is the following:

Theorem 1. Let \( p \) be a prime number and let \( K \) be the class of abelian groups \( A \) such that \( p^k A = 0 \). Then \( T(K) \) is undecidable.

An immediate consequence is

Corollary 2. \( T(\text{class of all abelian groups}) \) is undecidable.

Corollary 2 answers a few questions asked in [4].

1. Proof of Theorem 1. Let \( S \) be a finitely presented semigroup on two generators \( \alpha_1, \alpha_2 \) and defining relations \( V_\nu(\alpha_1, \alpha_2) = W_\nu(\alpha_1, \alpha_2)(\nu < n) \) such that \( S \) has undecidable word problem (see e.g. Davis [2]). We are going to define a finite extension \( T^* \) of \( T(K) \) and an effective map associating with every pair \( \langle V, W \rangle \) of words in \( \alpha_1, \alpha_2 \) an \( L \)-sentence \( \varphi \) such that \( V = W \) holds in \( S \) if and only if \( T^* \vdash \varphi \).

Let \( A \) be a \( p \)-group and \( a \in A \). Put \( \tau(a) = \langle h(a), h(pa), h(p^2a) \rangle \) where \( h \) is the \( p \)-height, i.e. \( h(x) = k \) if and only if \( x \in p^k A - p^{k+1} A \). Put \( \tau_0 = \langle 0, 2, 7 \rangle \), \( \tau_1 = \langle 0, 3, 6 \rangle \), \( \tau_2 = \langle 0, 4, 5 \rangle \). Note that for each pair \( \langle i, j \rangle \), \( i, j \leq 2 \), \( i \neq j \), \( \tau_i \) has a component which is greater than the corresponding component of \( \tau_j \).

For \( j = 1, 2 \) let \( \varphi_j(x, y) \) be the \( L \)-formula

\[
x = y = 0 \lor \exists x', y'(P(x') & P(y') & x = p^3x' & y = p^3y') \land \tau(x') = \tau_0 \land \tau(y') = \tau_j \land h(x' - y') = 1.
\]

Let \( T^* \) be the theory obtained from \( T(K) \) by adjoining axioms (i) and (ii) below.

Received by the editors March 26, 1975.

AMS (MOS) subject classifications (1970). Primary 02G05.

1 Supported by Schweizerischer Nationalfonds.

© American Mathematical Society 1976
\( \forall x(h(x) \geq 8 \rightarrow \exists ! y(h(y) \geq 8 \& \varphi_j(x,y))) \), \( j = 1, 2 \).

(i) simply says that \( \varphi_j \) defines a function \( f_j \) on the set of elements of height \( \geq 8 \). Therefore every word \( W(f_1, f_2) \) in \( f_1, f_2 \) defines a function \( \overline{W(f_1, f_2)} \), and a definition of this function can easily be written down in terms of \( \varphi_1, \varphi_2 \).

(ii) \( \forall x(h(x) \geq 8 \rightarrow V(f_1, f_2)(x) = \overline{W(f_1, f_2)}(x)) \), \( \nu < n \).

From (ii) it follows immediately that

\[
T^* \vdash \forall x(h(x) \geq 8 \rightarrow V(f_1, f_2)(x) = \overline{W(f_1, f_2)}(x))
\]

whenever \( V(a_1,a_2) = W(a_1,a_2) \) holds in \( S \). Since every countable semigroup can be embedded in the semigroup of endomorphisms of a countable vector space over the field \( F \) with \( p \) elements, the converse (and the theorem) clearly follow from

Claim. For any pair \( g_1, g_2 \) of endomorphisms of a countably infinite vectorspace \( V \) over \( F \) there exists a model \( \langle A, B \rangle \) of \( T(K) \) satisfying (i) such that \( \langle V, g_1, g_2 \rangle \cong \langle p^8 A, f_1, f_2 \rangle \) where \( f_1, f_2 \) are defined by \( \varphi_1, \varphi_2 \).

Proof of Claim. Put \( M = \{1, 3, 4, 6, 7, 9\} \) and for \( i \in M \) put \( A_i = (\mathbb{Z}/p^i\mathbb{Z})^\omega \), \( A = \bigoplus_{i \in M} A_i \), and let \( (a_{i,k})_{k \in \omega} \) be a basis of \( A_i \). Identify \( V \) with \( p^8 A = p^8 A_9 \) and let \( a_{9,k} \in A_9 \) such that \( g_j(p^8 a_{9,k}) = p^8 a_{9,\nu} \), \( j = 1, 2, k \in \omega \). For \( k \in \omega \) put

\[
\begin{align*}
b_{0,k} &= p^5 a_{9,k} + pa_{3,k} + a_{1,k} \\
b_{1,k} &= p^5 a_{9,k}^{(1)} + p^4 a_{4,k} + p^2 a_{4,k} + a_{1,k} \\
b_{2,k} &= p^5 a_{9,k}^{(2)} + p^3 a_{6,k} + a_{1,k}
\end{align*}
\]

and let \( B \) be the subgroup of \( A \) generated by all the \( b_{j,k} \)'s. Note that \( \tau(b_{j,k}) = \tau_j \). The important property of these generators is

\[
if b = \sum_{i \leq 2} \sum_{k \in \omega} \tau_{i,k} b_{i,k}, \tau_{i,k} \in \mathbb{Z}, \text{ and } \tau(b) = \tau_j, \text{ then}
\]

\[
(*) \quad p \text{ divides } \tau_{i,k} \text{ for all } (i,k) \text{ such that } i \neq j.
\]

Assume, e.g., \( \tau(b) = \tau_1 \). Then \( p \) divides \( \tau_{0,k} \) because otherwise \( h(pb) = 2 \), and \( p \) divides \( \tau_{2,k} \) because otherwise \( h(p^2 b) = 5 \). The remaining two cases are similar.

Next we show that \( \varphi_j(a,g_j(a)) \) holds in \( \langle A, B \rangle \) for all \( a \in p^8 A, j = 1, 2 \). This is clear if \( a = 0 \). Assume \( a = \sum_k \tau_k p^8 a_{9,k} \neq 0 \), \( 0 \leq \tau_k < p \). Put \( x' = \sum \tau_k b_{0,k}, y' = \sum \tau_k b_{j,k} \) and look at the definition of \( \varphi_j \) and \( B \).

It remains to show that \( \langle A, B \rangle \) satisfies (i). Assume \( \varphi_1(a,a_1) \) and \( \varphi_1(a,a_2) \) both hold in \( \langle A, B \rangle \) (the case \( j = 2 \) is analogous). To show: \( a_1 = a_2 \). Leaving the simpler case \( a = 0 \) to the reader we assume \( a \neq 0 \). By the definition of \( \varphi_1(x,y) \) there exist \( b_1', b_2', b_1, b_2 \in B \) such that for \( j = 1, 2 \)

\[
\begin{align*}
(1) & \quad a = p^3 b_j', \quad a_j = p^3 b_j, \\
(2) & \quad \tau(b_j') = \tau_0, \quad \tau(b_j) = \tau_1, \\
(3) & \quad h(b_j' - b_j) = 1.
\end{align*}
\]

Write \( b_j = \sum_{i \leq 2} \sum_{k \in \omega} \tau_{i,k} b_{i,k}, b_j = \sum_{i \leq 2} \sum_{k \in \omega} \tau_{i,k} b_{i,k}, \tau_{i,k}(j), \tau_{i,k}(j) \in \mathbb{Z} \); {\bullet \bullet}

and (2) imply

\[
(\ast) \quad p \text{ divides } \tau_{i,k} \text{ for all } (i,k) \text{ such that } i \neq j.
\]
(4) \( p \) divides \( r_{i,k}^{(j)}, r_{2,k}^{(j)}, s_{0,k}^{(j)}, s_{2,k}^{(j)} \) for all \( k \in \omega, j = 1, 2. \)

This together with (3) gives

(5) \( r_{0,k}^{(j)} = s_{1,k}^{(j)} \pmod{p} \) for all \( k \in \omega, j = 1, 2. \)

(1) and (4) imply

\[
a = \sum_k r_{0,k}^{(j)} p^3 b_{0,k}, \quad a_j = \sum_k s_{1,k}^{(j)} p^3 b_{1,k}, \quad j = 1, 2.
\]

Combining the last two equations with (5) we obtain \( s_{1,k}^{(1)} = s_{1,k}^{(2)} \pmod{p} \) for all \( k \), and therefore \( a_1 = a_2 \). This proves Theorem 1.

**Remark.** Although \( T(K) \) is undecidable it is impossible to interpret number theory in it. This is a consequence of the fact that \( T(K) \) is stable in the sense of Shelah [5] whereas number theory is unstable. Stability of \( T(K) \) follows from [1] and the proof of Corollary 3 below.

2. **Theories of modules.** For any recursive ring \( R \) with identity let \( T_R \) denote the first order theory of \( R \)-modules in the language with nonlogical symbols \( 0, +, f_r (r \in R) \) (cf. Eklof-Sabbagh [3]). Theorem 1 can be used to prove undecidability of \( T_R \) for various rings \( R \).

**Corollary 3.** There exist finite commutative rings \( R \) such that \( T_R \) is undecidable.

**Proof.** Put \( R = R'[X]/(X^2) \) where \( R' \) is the prime ring of characteristic \( 2^9 \). If \( M \) is an \( R \)-module then the pair \( \mathfrak{A}_M = \langle \langle m \in M | X m = 0 \rangle, XM \rangle \), considered as a pair of abelian groups, is a model of \( T(K) \). Conversely assume \( \langle A, B \rangle \models T(K) \). Let \( B_1 \) be an isomorphic copy of \( B \) and define an endomorphism \( X \) of \( M = A \oplus B_1 \) by \( X a = 0 \) for \( a \in A \), \( X b_1 = b \) for \( b_1 \in B_1 \). Clearly this provides \( M \) with an \( R \)-module structure, and \( \mathfrak{A}_M = \langle A, B \rangle \). This gives a faithful interpretation of \( T(K) \) in \( T_R \), hence \( T_R \) is undecidable.

Let \( F \) be a finite field. The decidability proof for the theory of abelian groups given by Szmielew [6] applies also to \( T_{F[X]} \). In contrast the next corollary shows that \( T_{F[X,Y]} \) is undecidable.

**Corollary 4.** If \( R \neq 0 \) is any recursive commutative ring then \( T_{R[X,Y]} \) is undecidable.

**Proof.** Replacing \( \rho \) by \( X \) and making a few obvious changes in the proof of Theorem 1 we obtain that the theory of the class of structures \( \langle M, N \rangle \), \( M \) an \( R[X] \)-module and \( N \subset M \) a submodule, is undecidable. By an argument similar to the one used in the proof of Corollary 3 it follows that \( T_{R[X,Y]} \) is undecidable.

**References**

4. G. T. Kuzlov and A. I. Kokorin, Elementary theory of abelian groups without torsion, with a


Seminar fur Angewandte Mathematik der Universitaet Zurich, Freierstrasse 36, 8032 Zurich, Switzerland