COMPACTNESS OF CERTAIN HOMOGENEOUS SPACES OF LOCALLY COMPACT GROUPS

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Abstract. Let $H$ be the fixed points of a family of automorphisms of a locally compact group $G$ with $G/H$ finite invariant measure. It is proved in this paper that when the 1-component of $G$ is open, $G/H$ is compact.

Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$ such that $G/H$ admits a finite $G$-invariant measure. Then $G/H$ is compact if $G$ is a connected Lie group and $H$ has finitely many connected components [6, Mostow], or if $G$ is a $p$-adic group and $H$ is discrete [8, Tamagawa]. Recently, Greenleaf-Moskowitz-Rothschild [1], [2] proved that $G/H$ is compact for disconnected Lie groups $G$ with $H$ consisting of the fixed points of a family of automorphisms of $G$ (see Lemma 3 below). Under similar restrictions on $H$ as in [1], the author [7] obtained the same result for linear algebraic groups defined over locally compact fields. Now in this paper, we prove the following theorem, which extends Lemma 3 to non-Lie groups.

Theorem. Let $G$ be a locally compact group and $H$ be a closed subgroup consisting of the fixed points of a family of automorphisms of $G$ such that $G/H$ has a finite $G$-invariant measure. If $G$ is $\alpha$-compact with its 1-component open then $G/H$ is compact. In particular, this is the case when $G$ is connected or when $G$ is $\alpha$-compact and locally connected.

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1. Preliminaries and notations. Throughout this paper we consider only $\sigma$-compact groups (i.e. groups which are countable union of compact subsets). For a locally compact group $G$, let $G_0$ denote its 1-component, $\mathfrak{A}(G)$ the group of topological automorphisms of $G$ and $\mathfrak{I}(G) = \{\alpha_x | x \in G\}$ the subgroup of inner automorphisms of $G$. Let $K(G_0)$ denote the maximal compact normal subgroup of $G_0$ (the existence of such a group is proved in [4]). For a subset $A$ of $\mathfrak{A}(G)$, let $G_A = \{g \in G | \alpha(g) = g, \alpha \in A\}$. It is easy to see that $G_A$ is a closed subgroup of $G$.

A locally compact space $X$ is called a homogeneous $G$-space if $G$ acts on $X$ transitively. Thus when $H$ is a closed subgroup of $G$, $G/H$ is a homogeneous $G$-space with the action of $G$ on $G/H$ by left translation. A regular Borel measure $m$ on $X$ is $G$-invariant if $m(gE) = m(E)$ for all $g \in G$ and all Borel subsets $E$ of $X$.

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**Lemma 1.** Let $G$ and $G'$ be two locally compact groups and $X$ (resp. $X'$) be a homogeneous $G$-space (resp. $G'$-space). Let $\pi: G \to G'$ be an open and continuous epimorphism and $\eta: X \to X'$ be a continuous surjection such that $\eta(gx) = \pi(g)\eta(x)$ $(g \in G, x \in X)$.

We have the following:

(i) $\eta$ is an open map.

(ii) If $X$ admits a finite $G$-invariant measure, then $X'$ admits a finite $G'$-invariant measure.

(iii) If $\eta$ is bijective (i.e. if the actions of $G$ on $X$ and of $G'$ on $X'$ are equivalent), then

(a) the converse of (ii) holds, and

(b) $X$ is compact if and only if $X'$ is compact.

**Proof.** (i) Let $U$ be a neighborhood of some $x$ in $X$; we show that $\eta$ maps $U$ to an open neighborhood of $\eta(x)$. Since $G$ and $G'$ are $\sigma$-compact, it follows that the mappings

$$G \to X, \quad G' \to X', \quad g \mapsto gx, \quad g' \mapsto g'\eta(x),$$

are both open and continuous (see e.g. [3, p. 7]). Let $V$ be an open neighborhood of $1$ in $G$ such that $Vx \subseteq U$. Hence $\pi(V)\eta(x) = \eta(Vx)$ is an open neighborhood of $\eta(x)$ contained in $\eta(U)$.

(ii) Let $m$ be a finite $G$-invariant measure on $X$. Define $m'$ on $X'$: $m'(E) = m(\eta^{-1}(E))$ for every Borel set $E$ of $X'$. Then $m'$ is a finite regular Borel measure on $X'$. Now for any $g' \in G'$, there exists a $g \in G$ such that $\pi(g) = g'$ and $\eta^{-1}(g'E) = g\eta^{-1}(E)$. Therefore

$$m'(g'E) = m(\eta^{-1}(g'E)) = m(g\eta^{-1}(E)) = m'(E).$$

(iii) (b) is obvious since $\eta$ is now a homeomorphism. For (a): Let $m'$ be a finite $G'$-invariant measure on $X'$ and define $m$ on $X$: $m(E) = \eta^{-1}(\eta(E))$ for any Borel subset $E$ of $X$. It is easy to see that $m$ is a finite $G$-invariant regular Borel measure.

**Lemma 2** [6, Lemma 2.5]. Let $H \subseteq F$ be closed subgroups of a locally compact group $G$ such that $G/H$ admits a finite invariant measure $m$. Then $G/F$ and $F/H$ admit finite invariant measures of which $m$ is a product.

**Lemma 3** [2, Theorem 2]. If $G$ is a Lie group and $m(G/G_A)$ is finite, then $G/G_A$ is compact.

**Lemma 4** [5, Theorem 2.3]. If $\pi: P \to P'$ is a continuous epimorphism of connected compact groups, then $\pi$ maps the center of $P$ onto the center of $P'$.

**2. Proof of the theorem.** First we show that it suffices to consider the case when $G$ is connected. $G_0$ is open so $G_0G_A$ is a closed subgroup of $G$. Hence, by Lemma 2, both $G_0/G_0G_A$ and $G_0G_A/G_A$ admit invariant measures. Since $G/G_0G_A$ is discrete, $G/G_0G_A$ is finite. On the other hand, the $G_0G_A$-space $G_0G_A/G_A$ is equivalent to the $G_0$-space $G_0G_0 \cap G_A$ and it follows from Lemma 1 that $G_0G_0 \cap G_A$ admits a finite invariant measure. But $G_0 \cap G_A$...
\((G_0)_A\) where \(A' = \{a_{G_0}|a \in A\}\) is a subset of \(\mathcal{A}(G_0)\). And so by assumption \(G_0/G_0 \cap G_A\) is compact and, by Lemma 1, \(G_0G_A/G_A\) is compact. Thus \(G/G_A\) compact follows. This completes the proof of the reduction to the case when \(G\) is connected.

From now on \(G\) is connected and we proceed to prove the theorem in four cases.

**Case (i).** \(K(G) = \{1\}\).

Let \(P\) be a compact normal subgroup of \(G\) such that \(G/P\) is a Lie group. Since \(PK(G)\) is a compact normal subgroup containing \(K(G)\) and \(K(G)\) is maximal, it follows that \(P \subset PK(G) = K(G) = \{1\}\). Therefore \(G\) is a Lie group. Thus by Lemma 3, \(G/G_A\) is compact.

**Case (ii).** \(K(G)_0 = \{1\}\).

Let \(G' = G/K(G)\) and \(\pi: G \to G'\) be the projection. For any \(\alpha \in \mathcal{A}(G)\), \(\alpha(K(G))\) is again a compact normal subgroup of \(G\) and so as in Case (i), \(\alpha(K(G)) \subset K(G)\) (i.e. \(K(G)\) is characteristic in \(G\)). Hence \(\alpha\) induces an automorphism \(\alpha'\) of \(G'\) such that for any \(g \in G\), \(\alpha'(\pi(g)) = \pi(\alpha(g))\). Let \(A' = \{\alpha'|a \in A\}\) and \(G'A' = \{g' \in G'|\alpha'(g') = g', \alpha' \in A'\}\). Define a mapping

\[
\eta: G/G_A \to G'/G'A', \quad \eta(gG_A) = \pi(g)G_A'.
\]

It is easy to see that \(\eta\) is a continuous surjection and so, by Lemma 1, \(G'/G'A'\) has a finite \(G'\)-invariant measure. Since the pull back of any compact normal subgroup of \(G'\) by \(\pi\) is a compact normal subgroup of \(G\) contained in \(K(G)\), it follows that \(K(G') = \{1\}\). Hence it follows from Case (i) that \(G'/G'A'\) is compact.

Let \(H_A = \{g \in G|\alpha(\alpha)g^{-1} \in K(G), \alpha \in A\}\). It is obvious that \(H_A = \pi^{-1}(G'_A)\) is a closed subgroup of \(G\). Define a mapping

\[
\psi: G/H_A \to G'/G'A', \quad \psi(gH_A) = \pi(g)G_A'.
\]

Then it is easy to see that \(\psi\) is a continuous bijection and so, by Lemma 1, \(G/H_A\) is compact. Hence for \(G/G_A\) to be compact, it remains to show that \(H_A/G_A\) is compact.

Since \(G_A \subset H_A\) and \(G/G_A\) has a finite \(G\)-invariant measure, it follows from Lemma 2 that \(H_A/G_A\) has a finite \(H_A\)-invariant measure. As \(G\) is connected, it is obvious that \(\mathfrak{g}_1(G)/K(G) \subset \mathfrak{k}(K(G)))_0\). Let \(\mathfrak{g}_1(K(G))_0\) denote the subgroup of inner automorphisms of \(K(G)\) induced by elements of \(K(G)\); then \(\mathfrak{k}(K(G))_0 = \mathfrak{g}_1(K(G))_0\) [4, Iwasawa]. Since \(K(G)_0 = \{1\}\), \([G,K(G)] = \{1\}\).

So for any \(g_1, g_2\) in \(H_A\), we have

\[
\alpha(g_1g_2)(g_1g_2)^{-1} = \alpha(g_1)(\alpha(g_2)g_2^{-1})g_1^{-1} = \alpha(g_1)g_1^{-1} \alpha(g_2)g_2^{-1}.
\]

Hence the mapping \(f: H_A \to K(G), f(g) = \alpha(g)g^{-1}\) is a homomorphism with \(G_A\) as its kernel. Hence \(G_A\) is normal in \(H_A\) and \(H_A/G_A\) is compact. This completes the proof of Case (ii).

**Case (iii).** The center \(Z\) of \(K(G)_0\) is trivial.

Since \(K(G)_0\) is characteristic in \(K(G)\) and \(K(G)\) is characteristic in \(G\), it follows that \(K(G)_0\) is characteristic in \(G\). Let \(G' = G/K(G)_0\) and \(\pi: G \to G'\) be the projection. Then as in Case (ii), \(A'\) induces a family \(A'\) of automorphisms
of $G'$ such that $G'/G'_A$ admits a finite $G'$-invariant measure. Since it is easy to see that $(K(G'))_0 = \{1\}$, it follows from Case (ii) that $G'/G'_A$ is compact.

Let $C = \{g \in G | gkg^{-1} = k, k \in K(G)_0\}$; then it follows from [4] that $G = K(G)_0 C$. Here $K(G)_0 \cap C = Z = \{1\}$ and $[K(G)_0, C] = \{1\}$. And so $G = K(G)_0 \times C$ is a direct product and $\pi_C: C \to G'$ is an isomorphism. Now let $C_A = \{c \in C | a(c) = c, a \in A\}$. We shall show that $\pi(C_A) = G'_A$. It is obvious that $\pi(C_A) \subset G'_A$. To see the converse inclusion, let $g' \in G'_A$; then there is a unique $c \in C$ such that $\pi(c) = g'$. So for any $a \in A$, $\pi(c) = \alpha'(\pi(c)) = \pi(a(c))$ or $\alpha(c)c^{-1} \in K(G)_0$. But $C$ is characteristic in $G$, hence $\alpha(c)c^{-1} \in C \cap K(G)_0$. Thus $\alpha(c) = c$ or $g' \in \pi(C_A)$. Therefore $\pi_C$ induces a homeomorphism $\eta: C/C_A \to G'/G'_A$, $\eta(cC_A) = \pi(c)G'_A$. Hence $C/C_A$ is compact.

Let $K_A = \{k \in K(G)_0 | a(k) = k, a \in A\}$ and define

$$\psi: (K(G)_0/K_A) \times (C/C_A) \to G/G_A, \psi(kK_A, cC_A) = kcG_A.$$

Here $\psi$ is well defined, for if $kk_1^{-1} \in K_A$, $cc_1^{-1} \in C_A$, then $(kc)(k_1c_1)^{-1} = (kk_1^{-1})(cc_1^{-1}) \in K_A C_A \subset G_A$.

Let $\psi_1: G \to G/G_A$ be the continuous projection and $\psi_2: K(G)_0 \times C \to (K(G)_0/K_A) \times (C/C_A)$ be the open projection. Then $\psi_1 = \psi \circ \psi_2$ since $G = K(G)_0 \times C$. Hence $\psi$ is continuous and $G/G_A$ is compact.

Case (iv). $Z \neq \{1\}$.

Since $K(G)_0$ is characteristic in $G$, so is $Z$. Let $G' = G/Z$ and $\pi: G \to G'$ be the projection. Then as in Case (ii) $A$ induces a family $A'$ of automorphisms of $G'$ such that $G'/G'_A$ admits a finite $G'$-invariant measure. As $Z$ is compact, therefore $\pi(K(G)_0) = (K(G'))_0$. Hence by Lemma 4, we have center of $(K(G'))_0 = \pi(Z)$. But $\pi(Z) = \{1\}$, therefore it follows from Case (iii) that $G'/G'_A$ is compact.

Let $H_A = \{g \in G | a(g)g^{-1} \in Z, a \in A\}$. Then $H_A = \pi^{-1}(G'_A)$ is a closed subgroup of $G$ containing $G_A$. As in Case (ii), we have $G/H_A$ compact. So for $G/G_A$ to be compact, it remains to prove that $H_A/G_A$ is compact. Now

$$\exists(G)_Z \subset \forall(Z)_0 = \exists^*(Z_0) = \{1\}$$

where $\exists^*(Z_0)$ denotes the subgroup of inner automorphisms of $Z$ induced by elements of $Z_0$. Therefore $[Z, \bar{G}] = \{1\}$ and analogous arguments as those in Case (ii) show that $H_A/G_A$ is compact. This completes the proof of the theorem.

REFERENCES


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