COMPACTNESS OF CERTAIN HOMOGENEOUS SPACES OF LOCALLY COMPACT GROUPS

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Abstract. Let $H$ be the fixed points of a family of automorphisms of a locally compact group $G$ with $G/H$ finite invariant measure. It is proved in this paper that when the 1-component of $G$ is open, $G/H$ is compact.

Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$ such that $G/H$ admits a finite $G$-invariant measure. Then $G/H$ is compact if $G$ is a connected Lie group and $H$ has finitely many connected components [6, Mostow], or if $G$ is a $p$-adic group and $H$ is discrete [8, Tamagawa]. Recently, Greenleaf-Moskowitz-Rothschild [1], [2] proved that $G/H$ is compact for disconnected Lie groups $G$ with $H$ consisting of the fixed points of a family of automorphisms of $G$ (see Lemma 3 below). Under similar restrictions on $H$ as in [1], the author [7] obtained the same result for linear algebraic groups defined over locally compact fields. Now in this paper, we prove the following theorem, which extends Lemma 3 to non-Lie groups.

Theorem. Let $G$ be a locally compact group and $H$ be a closed subgroup consisting of the fixed points of a family of automorphisms of $G$ such that $G/H$ has a finite $G$-invariant measure. If $G$ is $\sigma$-compact with its 1-component open then $G/H$ is compact. In particular, this is the case when $G$ is connected or when $G$ is $\sigma$-compact and locally connected.

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Lemma 1. Let $G$ and $G'$ be two locally compact groups and $X$ (resp. $X'$) be a homogeneous $G$-space (resp. $G'$-space). Let $\pi: G \to G'$ be an open and continuous epimorphism and $\eta: X \to X'$ be a continuous surjection such that $\eta(gx) = \pi(g)\eta(x)$ ($g \in G, x \in X$).

We have the following:

(i) $\eta$ is an open map.

(ii) If $X$ admits a finite $G$-invariant measure, then $X'$ admits a finite $G'$-invariant measure.

(iii) If $\eta$ is bijective (i.e. if the actions of $G$ on $X$ and of $G'$ on $X'$ are equivalent), then

(a) the converse of (ii) holds, and

(b) $X$ is compact if and only if $X'$ is compact.

Proof. (i) Let $U$ be a neighborhood of some $x$ in $X$; we show that $\eta(U)$ contains an open neighborhood of $\eta(x)$. Since $G$ and $G'$ are $\sigma$-compact, it follows that the mappings

$$G \to X, \quad G' \to X',
$$

$$g \mapsto gx, \quad g' \mapsto g'\eta(x),$$

are both open and continuous (see e.g. [3, p. 7]). Let $V$ be an open neighborhood of 1 in $G$ such that $Vx \subseteq U$. Hence $\pi(V)\eta(x) = \eta(Vx)$ is an open neighborhood of $\eta(x)$ contained in $\eta(U)$.

(ii) Let $m$ be a finite $G$-invariant measure on $X$. Define $m'$ on $X'$: $m'(E) = m(\eta^{-1}(E))$ for every Borel set $E$ of $X'$. Then $m'$ is a finite regular Borel measure on $X'$. Now for any $g' \in G'$, there exists a $g \in G$ such that $\pi(g) = g'$ and $\eta^{-1}(g'E) = g\eta^{-1}(E)$. Therefore

$$m'(g'E) = m(\eta^{-1}(g'E)) = m(g\eta^{-1}(E)) = m'(E).$$

(iii) (b) is obvious since $\eta$ is now a homeomorphism. For (a): Let $m'$ be a finite $G'$-invariant measure on $X'$ and define $m$ on $X$: $m(E) = m'(\eta(E))$ for any Borel subset $E$ of $X$. It is easy to see that $m$ is a finite $G$-invariant regular Borel measure.

Lemma 2 [6, Lemma 2.5]. Let $H \subset F$ be closed subgroups of a locally compact group $G$ such that $G/H$ admits a finite invariant measure $m$. Then $G/F$ and $F/H$ admit finite invariant measures of which $m$ is a product.

Lemma 3 [2, Theorem 2]. If $G$ is a Lie group and $m(G/G_A)$ is finite, then $G/G_A$ is compact.

Lemma 4 [5, Theorem 2.3]. If $\pi: P \to P'$ is a continuous epimorphism of connected compact groups, then $\pi$ maps the center of $P$ onto the center of $P'$.

2. Proof of the theorem. First we show that it suffices to consider the case when $G$ is connected. $G_0$ is open so $G_0G_A$ is a closed subgroup of $G$. Hence, by Lemma 2, both $G/G_0G_A$ and $G_0G_A/G_A$ admit invariant measures. Since $G/G_0G_A$ is discrete, $G/G_0G_A$ is finite. On the other hand, the $G_0G_A$-space $G_0G_A/G_A$ is equivalent to the $G_0$-space $G_0/G_0 \cap G_A$ and it follows from Lemma 1 that $G_0G_0 \cap G_A$ admits a finite invariant measure. But $G_0 \cap G_A$
\( (G_0)_A \) where \( A' = \{ \alpha_{(g_0)} \mid \alpha \in A \} \) is a subset of \( \mathfrak{A}(G_0) \). And so by assumption \( \overline{G_0} / G_0 \cap G_0 \) is compact and, by Lemma 1, \( G_0 G_0 / G_0 \) is compact. Thus \( G / G_0 \) compact follows. This completes the proof of the reduction to the case when \( G \) is connected.

From now on \( G \) is connected and we proceed to prove the theorem in four cases.

Case (i). \( K(G) = \{1\} \).

Let \( P \) be a compact normal subgroup of \( G \) such that \( G/P \) is a Lie group. Since \( PK(G) \) is a compact normal subgroup containing \( K(G) \) and \( K(G) \) is maximal, it follows that \( P \subset PK(G) = K(G) = \{1\} \). Therefore \( G \) is a Lie group. Thus by Lemma 3, \( G/\mathfrak{A} \) is compact.

Case (ii). \( K(G)_0 = \{1\} \).

Let \( G' = G/K(G) \) and \( \pi: G \to G' \) be the projection. For any \( \alpha \in \mathfrak{A}(G) \), \( \alpha(K(G)) \) is again a compact normal subgroup of \( G \) and so as in Case (i), \( \alpha(K(G)) \subset K(G) \) (i.e. \( K(G) \) is characteristic in \( G \)). Hence \( \alpha \) induces an automorphism \( \alpha' \) of \( G' \) such that for any \( g \in G, \alpha'(\pi(g)) = \pi(\alpha(g)) \). Let \( A' = \{ \alpha' \mid \alpha \in A \} \) and \( G'A' = \{ g' \in G' \mid \alpha'(g') = g', \alpha' \in A' \} \). Define a mapping

\[ \eta: G/\mathfrak{A} \to G'/G'A', \quad \eta(gG_0) = \pi(g)G_0A'. \]

It is easy to see that \( \eta \) is a continuous surjection and so, by Lemma 1, \( G'/G'A' \) has a finite \( G' \)-invariant measure. Since the pull back of any compact normal subgroup of \( G' \) by \( \pi \) is a compact normal subgroup of \( G \) contained in \( K(G) \), it follows that \( K(G') = \{1\} \). Hence it follows from Case (i) that \( G'/G'A' \) is compact.

Let \( H_A = \{ g \in G \mid \alpha(g)g^{-1} \in K(G), \alpha \in A \} \). It is obvious that \( H_A = \pi^{-1}(G'A') \) is a closed subgroup of \( G \). Define a mapping

\[ \psi: G/H_A \to G'/G'A', \quad \psi(gH_A) = \pi(g)G_0A'. \]

Then it is easy to see that \( \psi \) is a continuous bijection and so, by Lemma 1, \( G/H_A \) is compact. Hence for \( G/\mathfrak{A} \) to be compact, it remains to show that \( H_A/\mathfrak{A} \) is compact.

Since \( \mathfrak{A} \subset H_A \) and \( G/\mathfrak{A} \) has a finite \( G \)-invariant measure, it follows from Lemma 2 that \( H_A/\mathfrak{A} \) has a finite \( H_A \)-invariant measure. As \( G \) is connected, it is obvious that \( \mathfrak{I}(G)_{K(G)} \subset \mathfrak{I}(K(G))_0 \). Let \( \mathfrak{I}^*(K(G)_0) \) denote the subgroup of inner automorphisms of \( K(G) \) induced by elements of \( K(G)_0 \); then \( \mathfrak{I}(K(G))_0 = \mathfrak{I}^*(K(G)_0) \); [4, Iwasawa]. Since \( K(G)_0 = \{1\}, [G, K(G)] = \{1\} \). So for any \( g_1, g_2 \) in \( H_A \), we have

\[ \alpha(g_1g_2)(g_1^{-1}g_2) = \alpha(g_1)(\alpha(g_2)g_2^{-1})g_1^{-1} = \alpha(g_1)g_1^{-1} \alpha(g_2)g_2^{-1}. \]

Hence the mapping \( f: H_A \to K(G), f(g) = \alpha(g)g^{-1} \) is a homomorphism with \( G_A \) as its kernel. Hence \( G_A \) is normal in \( H_A \) and \( H_A/G_A \) is compact. This completes the proof of Case (ii).

Case (iii). The center \( Z \) of \( K(G)_0 \) is trivial.

Since \( K(G)_0 \) is characteristic in \( K(G) \) and \( K(G) \) is characteristic in \( G \), it follows that \( K(G)_0 \) is characteristic in \( G \). Let \( G' = G/K(G)_0 \) and \( \pi: G \to G' \) be the projection. Then as in Case (ii), \( A' \) induces a family \( A' \) of automorphisms

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of $G'$ such that $G'/G'_A$ admits a finite $G'$-invariant measure. Since it is easy to see that $(K(G'))_0 = \{1\}$, it follows from Case (ii) that $G'/G'_A$ is compact.

Let $C = \{ g \in G | gkg^{-1} = k, k \in (K(G))_0 \}$; then it follows from [4] that $G = K(G)_0C$. Here $K(G)_0 \cap C = Z = \{1\}$ and $[K(G)_0, C] = \{1\}$. And so $G = K(G)_0 \times C$ is a direct product and $\pi_C : C \to G'$ is an isomorphism. Now let $C_A = \{ c \in C | \alpha(c) = c, \alpha \in A \}$. We shall show that $\pi(C_A) = G'_A$. It is obvious that $\pi(C_A) \subset G'_A$. To see the converse inclusion, let $g' \in G'_A$; then there is a unique $c \in C$ such that $\pi(c) = g'$. So for any $\alpha \in A$, $\pi(c) = \alpha'(\pi(c)) = \pi(\alpha(c))$ or $\alpha(c)c^{-1} \in K(G)_0$. But $C$ is characteristic in $G$, hence $\alpha(c)c^{-1} \subset C \cap K(G)_0$. Thus $\alpha(c) = c$ or $g' \in \pi(C_A)$. Therefore $\pi_C$ induces a homeomorphism $\eta : C/C_A \to G'/G'_A$, $\eta(cC_A) = \pi(c)G'_A$. Hence $C/C_A$ is compact.

Let $K_A = \{ k \in K(G)_0 | \alpha(k) = k, \alpha \in A \}$ and define

$$\psi : (K(G)_0/K_A) \times (C/C_A) \to G/G_A, \psi(kK_A, cC_A) = kcG_A.$$ 

Here $\psi$ is well defined, for if $kk_1^{-1} \in K_A$, $cc_1^{-1} \in C_A$, then $(kc)(kc_1)^{-1} = (kk_1^{-1})(cc_1^{-1}) \in K_A C_A \subset G_A$.

Let $\psi_1 : G \to G/G_A$ be the continuous projection and $\psi_2 : K(G)_0 \times C \to (K(G)_0/K_A) \times (C/C_A)$ be the open projection. Then $\psi_1 = \psi \circ \psi_2$ since $G = K(G)_0 \times C$. Hence $\psi$ is continuous and $G/G_A$ is compact.

Case (iv). $Z \neq \{1\}$.

Since $(K(G)_0)_Z$ is characteristic in $G$, so is $Z$. Let $G' = G/Z$ and $\pi : G \to G'$ be the projection. Then as in Case (ii) $A$ induces a family $A'$ of automorphisms of $G'$ such that $G'/G'_A$ admits a finite $G'$-invariant measure. As $Z$ is compact, therefore $\pi((K(G)_0) = (K(G'))_0$. Hence by Lemma 4, we have center of $(K(G'))_0 = \pi(Z)$. But $\pi(Z) = \{1\}$, therefore it follows from Case (iii) that $G'/G'_A$ is compact.

Let $H_A = \{ g \in G | \alpha(g)g^{-1} \in Z, \alpha \in A \}$. Then $H_A = \pi^{-1}(G_A)$ is a closed subgroup of $G$ containing $G_A$. As in Case (ii), we have $G/H_A$ compact. So for $G/G_A$ to be compact, it remains to prove that $H_A/G_A$ is compact. Now

$$\mathfrak{Z}(G)_Z \subset \mathfrak{H}(Z)_0 = \mathfrak{Z}^*(Z_0) = \{1\}$$

where $\mathfrak{Z}^*(Z_0)$ denotes the subgroup of inner automorphisms of $Z$ induced by elements of $Z_0$. Therefore $[Z, G] = \{1\}$ and analogous arguments as those in Case (ii) show that $H_A/G_A$ is compact. This completes the proof of the theorem.

References


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