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A NOTE ON SEMILOCAL RINGS

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ABSTRACT. If $(R, m_1, \ldots, m_w)$ is a semilocal ring whose ideal lattice is topologically complete, it is shown that: given any natural number n and any decreasing sequence $\langle a_i \rangle$ of ideals of R, there exists a natural number $s(n)$ such that $a_{s(n)} \subseteq (\bigcap_i a_i) + m^n$ where $m = \bigcap_{i=1}^{w} m_i$. This generalizes a well-known theorem on complete semilocal rings.

The following is a well-known property of Noetherian semilocal rings (cf. [4, (30.1), p. 103] and [5, Theorem 13, p. 270]): if $(R, m_1, \ldots, m_w)$ is a Noetherian semilocal ring, $m = \bigcap_{i=1}^{w} m_i$, and R is complete in the m-adic topology, then, given a decreasing sequence $\langle a_i \rangle$ of ideals of R with $\bigcap_i a_i = 0$ and a natural number n, there exists a natural number $s(n)$ such that $a_{s(n)} \subseteq m^n$. The purpose of this note is to show that this result is a consequence of a more general theorem and that the conclusion depends in fact on the topological completeness of the ideal lattice of R and not on the completeness of R.

We adopt the ring terminology of [4] (in particular semilocal means Noetherian) and fix our notation as follows: $(R, m_1, \ldots, m_w)$ is a semilocal ring, $m = \bigcap_{i=1}^{w} m_i$, and $L(R)$ is the Noether lattice of ideals of R. Define a metric (called the m-adic metric) on $L(R)$ by

$$d(a, b) = 2^{-s(a,b)} \quad \text{where } s(a, b) = \sup\{n \mid a + m^n = b + m^n\}.$$

This metric gives rise to the m-adic completion, $L(R)^*$, of $L(R)$ which is a semilocal Noether lattice [1, Theorem 6.2, p. 200].

THEOREM 1. *Let $L(R)$ be complete in the m-adic topology. Given a decreasing sequence $\langle a_i \rangle$ of ideals of R and a natural number n, there exists a natural number $s(n)$ such that $a_{s(n)} \subseteq (\bigcap_i a_i) + m^n$.*

PROOF. Since $\langle a_i \rangle$ is decreasing and R/m^n satisfies the d.c.c. [4, Proof of 30.1, p. 103], $\langle a_i \rangle$ is Cauchy, and thus there exists $c \in L(R)$ such that $a_i \to c$ as $i \to \infty$. Now, let $j \geqslant 1$ be an integer and note that, since $a_i \to c$, for each integer $k \geqslant 0$, there exists an integer $i(k) \geqslant 0$ such that $c \subseteq c + m^k = a_{i(k)} + m^k \subseteq a_j + m^k$ and so $c \subseteq \bigcap_k (a_j + m^k) = a_j$. It follows that $c \subseteq \bigcap_i a_i$. Since $a_i \to c$, there is an integer $s(n)$ such that

$$a_{s(n)} \subseteq a_{s(n)} + m^n = c + m^n \subseteq (\bigcap_i a_i) + m^n$$

which completes the proof.

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THEOREM 2. *If R is complete in the m-adic topology, then $L(R)$ is complete in the m-adic topology.*

PROOF. $L(R) \cong L(R^*) \cong L(R)^*$ [3, Theorem 5] and $L(R)^*$ is a complete semilocal Noether lattice with its semilocal topology.

Theorems 1 and 2 yield

COROLLARY 3. *Let R be complete in the m-adic topology. Given a decreasing sequence $\langle a_i \rangle$ of ideals of R and a natural number n, there exists a natural number $s(n)$ such that $a_{s(n)} \subseteq (\bigcap_i a_i) + m^n$.*

Theorem 2 shows that the class of topologically complete semilocal rings is contained in the class of semilocal rings whose ideal lattice is topologically complete. This containment is proper as shown by the following example. Let Z be the integers and $p \in Z$ be prime. Then Z_{pZ} is not complete, but $L(Z_{pZ})$ is complete (by [2, Theorem 1, p. 197] the ideal lattice of every discrete valuation ring is complete).

REFERENCES

1. E. W. Johnson and J. A. Johnson, *Lattice modules over semi-local Noether lattices*, Fund. Math. **68** (1970), 187–201. MR **42** #4536.

2. E. W. Johnson, *A note on quasi-complete local rings*, Colloq. Math. **21** (1970), 197–198. MR **42** #262.

3. J. A. Johnson, *Semi-local lattices*, Fund. Math. (to appear).

4. M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR **27** #5790.

5. O. Zariski and P. Samuel, *Commutative algebra*. Vol. II, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #11006.

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