AN ANALOGUE OF SOME INEQUALITIES OF P. TURAN
CONCERNING ALGEBRAIC POLYNOMIALS
HAVING ALL ZEROS INSIDE \([-1, +1]\)

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Abstract. Let \(P_n(x)\) be an algebraic polynomial of degree \(< n\) having all its zeros inside \([-1, +1]\); then we have
\[
\int_{-1}^{1} P_n^2(x) \, dx > \frac{n}{2} \int_{-1}^{1} P_n^2(x) \, dx.
\]
The result is essentially best possible. Other related results are also proved.

Let \(H_n\) be the set of all polynomials whose degree does not exceed \(n\), i.e., polynomials of the form
\[
P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.
\]
Here the coefficients \(c_0, c_1, \ldots, c_n\) are arbitrary real numbers. The following inequalities on algebraic polynomials are well known.

**Theorem A.** Let \(P_n(x) \in H_n\); then we have
\[
\max_{-1 < x < 1} (1 - x^2) P_n^2(x) \leq n^2 \max_{-1 < x < 1} P_n^2(x),
\]
and
\[
\max_{-1 < x < 1} |P_n'(x)| \leq n^2 \max_{-1 < x < 1} |P_n(x)|.
\]
\((P_n'(x)\) stands for the derivative of \(P_n(x)\).)

(1.1) is due to S. N. Bernstein [1] and (1.2) to A. A. Markov [2]. In this work we are concerned with the following beautiful theorem of P. Turán [4].

**Theorem B.** Let \(P_n(x)\) be an algebraic polynomial of degree \(< n\) having all its zeros inside \([-1, +1]\); then we have
\[
\max_{-1 < x < 1} |P_n'(x)| \geq \frac{n^{1/2}}{6} \max_{-1 < x < 1} |P_n(x)|.
\]

(1.3) was later sharpened by Janos Eröd [2], who proved

**Theorem C.** Under the assumptions of Theorem B we have for \(P_n(x) \in H_n\)
Further, this result is best possible.

Inequalities on polynomials analogous to (1.2) in the norm
\[
\|f\|_{L^2[-1, +1]}^2 = \int_{-1}^{1} f^2(x) \, dx
\]
were proved by E. Schmidt [5]. See also the contributions of Einar Hille, G. Szegö and J. D. Tamarkin [4].

**Theorem D [E. Schmidt].** Let us denote
\[
M_n^2 = \max_{f \in H_n} \left[ \frac{\int_{-1}^{+1} f^2(x) \, dx}{\int_{-1}^{+1} f^2(x) \, dx} \right];
\]
then for \( n > 5 \) \((-6 < R < 13)\)
\[
M_n = \left( \frac{n + \frac{3}{2}}{\pi} \right)^2 \left( 1 - \frac{\pi^2 - 3}{12(n + \frac{3}{2})^2} + \frac{R}{(n + \frac{3}{2})^4} \right)^{-1}.
\]

In view of Theorems B and D it is natural to ask: If \( P_n \in H_n \) and all zeros of \( P_n(x) \) are inside \([-1, +1]\), then how small can the expression \( \int_{-1}^{+1} P_n^2(x) \, dx / \int_{-1}^{+1} f^2(x) \, dx \) be? The following theorem concerns the above question.

**Theorem 1.** Let \( P_n(x) \in H_n \) and assume that all its zeros are inside \([-1, +1]\); then we have
\[
\int_{-1}^{+1} P_n^2(x) \, dx > \frac{n}{2} \int_{-1}^{+1} P_n^2(x) \, dx.
\]
This result is best possible in the sense that there exists a polynomial \( p_0(x) \) of degree \( n \) having all zeros inside \([-1, +1]\) and for which
\[
\int_{-1}^{+1} P_0^2(x) \, dx / \int_{-1}^{+1} P_0^2(x) \, dx = \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n - 1)}, \quad n > 1.
\]
The proof of Theorem 1 is based on

**Theorem 2A.** Let \( f_n(x) \) be an algebraic polynomial of degree \( \leq n \) having all zeros inside \([-1, +1]\); then we have
\[
\frac{n}{2} \leq \int_{-1}^{+1} f_n^2(x)(1 - x^2) \, dx / \int_{-1}^{+1} f_n^2(x) \, dx,
\]
with equality only for \( f_n(x) = (1 + x)^p(1 - x)^q, \quad p + q = n \).
**Theorem 2B.** Let \( f_n(x) \) be an algebraic polynomial of degree \( \leq n \); then we have

\[
\int_{-1}^{+1} f_n^2(x) (1 - x^2) \, dx \bigg/ \int_{-1}^{+1} f_n^2(x) \, dx \leq n(n + 1),
\]

with equality only for \( f_n(x) = cP_n(x) \) (\( P_n(x) \) being the Legendre polynomial of degree \( n \)).

(1.10) may be regarded as analogous to (1.1) in the \( L_2 \) norm.

2. Since \( f_n(x) \) has all zeros inside \([-1, +1]\) we may write

\[
f_n(x) = \prod_{k=1}^{n} (x - x_k)
\]

where \(-1 \leq x_k \leq +1, k = 1, 2, \ldots, n\).

Professor P. Turán [6] observed that

\[
f_n'(x) = f_n(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)}
\]

and

\[
f_n^2(x) - f_n(x) f_n''(x) = f_n^2(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)^2}.
\]

On multiplying (2.3) by \( (1 - x^2) \) and using (2.2) we obtain

\[
2(1 - x^2)f_n^2(x) - \frac{d}{dx} \left( (1 - x^2)f_n(x)f_n'(x) \right)
\]

\[
= (1 - x^2)f_n^2(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)^2} + 2xf_n'(x)f_n''(x)
\]

\[
= f_n^2(x) \sum_{k=1}^{n} \frac{(1 - x^2) + 2x(x - x_k)}{(x - x_k)^2}.
\]

Therefore

\[
2(1 - x^2)f_n^2(x) - nf_n^2(x) - \frac{d}{dx} \left\{ (1 - x^2)f_n(x)f_n'(x) \right\}
\]

\[
= f_n^2(x) \sum_{k=1}^{n} \frac{1 - x^2 + 2x(x - x_k) - (x - x_k)^2}{(x - x_k)^2}
\]

\[
= f_n^2(x) \sum_{k=1}^{n} \frac{(1 - x_k^2)}{(x - x_k)^2} \geq 0.
\]

On integrating both sides from \(-1\) to \(1\) we obtain

\[
2 \int_{-1}^{+1} (1 - x^2)f_n^2(x) \, dx - n \int_{-1}^{+1} f_n^2(x) \, dx \geq 0,
\]

with equality only for \( f_n(x) = (1 + x)^p(1 - x)^q, p + q = n \). But this in turn implies (1.9). This proves Theorem 2A.

The proof of Theorem 2B depends on...
\[ \int_{-1}^{+1} p_i(x) p_j(x) \, dx = 0, \quad i \neq j, \]
\[ = \frac{2}{2i+1}, \quad i = j, \]
and
\[ \int_{-1}^{+1} (1 - x^2) p_i'(x) p_j'(x) \, dx = 0, \quad i \neq j, \]
\[ = \frac{2i(i+1)}{2i+1}, \quad i = j, \]
where \( p_j(x) \) denotes the Legendre polynomial of degree \( j \) in \( x \). Writing
\[ f_n(x) = \sum_{i=0}^{n} \lambda_i p_i(x), \quad f_n'(x) = \sum_{i=1}^{n} \lambda_i p_i'(x) \]
and using (2.6) and (2.7) we obtain
\[ \frac{\int_{-1}^{+1} f_n^2(x)(1 - x^2) \, dx}{\int_{-1}^{+1} f_n^2(x) \, dx} = \frac{\sum_{i=0}^{n} \lambda_i^2(i+1)}{\sum_{i=0}^{n} \lambda_i^2/2(i+1)} \leq n(n+1). \]
Obviously equality occurs only if \( f_n(x) = cP_n(x) \) (\( P_n(x) \) being Legendre polynomial of degree \( n \)). This proves Theorem 2B.

PROOF OF THEOREM 1. Since \( 1 - x^2 \leq 1, -1 < x < +1 \), we have
\[ (1 - x^2)P_n^2(x) \leq P_n^2(x) \quad \text{for} \quad -1 < x < +1. \]
Therefore
\[ \int_{-1}^{+1} (1 - x^2)P_n^2(x) \, dx \leq \int_{-1}^{+1} P_n^2(x) \, dx. \]
But from Theorem 2A we have
\[ \int_{-1}^{+1} (1 - x^2)P_n^2(x) \, dx \geq \frac{n}{2} \int_{-1}^{+1} P_n^2(x) \, dx. \]
From (2.8) and (2.9) it follows that
\[ \int_{-1}^{+1} P_n^2(x) \, dx \geq \frac{n}{2} \int_{-1}^{+1} P_n^2(x) \, dx. \]
This proves (1.7). It remains to prove (1.8). Let \( f_n(x) = p_0(x) = (1 - x^2)^m \), \( 2m = n \); then
\[ \int_{-1}^{+1} p_0^2(x) \, dx = \int_{-1}^{+1} (1 - x^2)^m \, dx = \frac{2\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \]
and
\[ \int_{-1}^{+1} P_0^2(x) \, dx = n^2 \int_{-1}^{+1} (1 - x^2)^{n-2} \, dx \]
\[ = 4n^2 \int_{0}^{\pi/2} \sin^{2n-3} \theta \cos^2 \theta \, d\theta \]
\[ = \frac{n^2 \Gamma(n-1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}. \]
Therefore

\[
\frac{\int_{-1}^{1} P_0^2(x) \, dx}{\int_{-1}^{1} P_0(x) \, dx} = n \left( n + \frac{1}{2} \right) \frac{n}{2(n - 1)} = \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n - 1)}, \quad n > 1.
\]

This proves Theorem 1 as well.

REFERENCES


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