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EXISTENCE OF SIDON SETS IN DISCRETE *FC*-GROUPS

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ABSTRACT. In this paper we prove that every infinite subset of a discrete *FC*-group contains an infinite Sidon set.

1. **Introduction.** Let G be an infinite discrete amenable group and $B(G)$ the Fourier-Stieltjes algebra of G (we refer to Eymard [2] for notations and properties). A subset $E \subseteq G$ is called a Sidon set if, for every bounded complex-valued function g on E, there is a function $f \in B(G)$ such that $f(x) = g(x)$ whenever $x \in E$.

When G is abelian with dual group Γ, $B(G)$ consists of those complex functions which are Fourier-Stieltjes transforms of measures in $M(\Gamma)$, so that the above definition coincides, for abelian groups, with the usual one. For further properties we refer to [1], [3] and [4] where Sidon sets in nonamenable groups are also discussed. It is well known that every infinite subset of an abelian group contains an infinite Sidon set: whether this is true for amenable noncommutative groups is still an open question. For certain groups, e.g. type I groups [8, Theorem 6] or solvable groups [5], the problem has an affirmative trivial answer; this is a consequence of the corresponding property for the commutative groups and of functorial properties of $B(G)$ [2, 2.31 and 2.36].

In this paper we prove that every infinite subset of G contains an infinite Sidon set when G is an *FC*-group, i.e. a group with finite conjugacy classes. Our proof follows from an application, suggested to us by A. Figà-Talamanca, of a general result of H. P. Rosenthal [6] on the l^1-subspaces of a Banach space. Notice that, unlike commutative groups, Riesz products techniques do not seem to work well in nonabelian groups; we refer to Cygan [1] for a study of Riesz products in *FC*-groups.

2. **Existence of Sidon sets.** Let $G_1, G_2, \ldots, G_i, \ldots$ be a sequence of finite groups and denote by $G^* = \prod_{i=1}^{\infty *} G_i$ their weak direct product endowed with the discrete topology. For every element $y \in G^*$ we denote by $y^{(i)}$ the i coordinate, and by e_i the identity of G_i; then $y^{(i)} \neq e_i$ only for finitely many i's. If y is not the identity of G^*, $\nu(y)$ will denote the largest index ν such that $y^{(\nu)} \neq e_\nu$. Finally, for every $y^{(i)}$, let $H(y^{(i)})$ denote the cyclic subgroup of G_i generated by $y^{(i)}$.

BASIC LEMMA. *For every infinite sequence $(y_n) \subseteq G^*$ there is a positive definite function f such that $f(y_n)$ does not converge as $n \to \infty$.*

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PROOF. We may suppose $y_n \neq y_m$ if $n \neq m$. Let $\|y_n^{(i)}\|$ be the infinite matrix whose rows are the coordinates of the y_n's. Let i_1 be the first column containing an element $y_{\bar{n}}^{(i_1)} \neq e_{i_1}$. Since G_{i_1} is finite there is $a_1 \in G_{i_1}$ appearing infinitely many times in the i_1th column. If $a_1 = e_{i_1}$, we let $n_1 = \bar{n}$; otherwise we let n_1 be the smallest n such that $a_1 = y_n^{(i_1)}$. Hence, for infinitely many rows, $y_n^{(i_1)} \in H(y_{n_1}^{(i_1)})$. We select these rows and form the infinite submatrix whose rows have been just chosen and whose column index starts from $\nu(y_{n_1})$. We choose as before i_2 and n_2; hence, for infinitely many n's, we have at same time $y_n^{(i_1)} \in H(y_{n_1}^{(i_1)})$ and $y_n^{(i_2)} \in H(y_{n_2}^{(i_2)})$. Carrying on this process we produce two infinite subsequences (i_k) and (n_k) with the following properties:

(1) $y_{n_k}^{(i_k)} \neq e_{i_k}$.

(2) $i_k > \nu(y_{n_{k-1}}) \geqslant i_{k-1}$.

(3) $y_{n_r}^{(i_k)} \in H(y_{n_k}^{(i_k)}) = H_{i_k}$ whenever $r \geqslant k$.

Let $\gamma^{(i_1)}$ be an arbitrary character of H_{i_1}. Suppose that $\gamma^{(i_1)}, \ldots, \gamma^{(i_k)}$ have been chosen, $\gamma^{(i_s)} \in \hat{H}_{i_s}$, where $\hat{H}_{i_s}$ is the dual group of H_{i_s}, and $s = 1, \ldots, k$. We define $\gamma^{(i_{k+1})} \in \hat{H}_{i_{k+1}}$ to be the identity if

$$\left| \prod_{s=1}^{k} \gamma^{(i_s)}(y_{n_k}^{(i_s)}) - \prod_{s=1}^{k} \gamma^{(i_s)}(y_{n_{k+1}}^{(i_s)}) \right| \geqslant \frac{1}{2}$$

(this product makes sense by (3)); otherwise, we choose, on account of (1), $\gamma^{(i_{k+1})}$ in such a way that $\pi \geqslant \arg(\gamma^{(i_{k+1})}(y_{n_{k+1}}^{(i_{k+1})})) > \pi/3$. Let $g^{(i_k)}$ be a positive definite function on G_{i_k} such that $g^{(i_k)}(x) = \gamma^{(i_k)}(x)$ for every $x \in H_{i_k}$. Let f_{i_k} be a positive-definite function on the whole of G^* such that $f_{i_k}(y) = g^{(i_k)}(y^{(i_k)})$ for every $y \in G^*$. Finally, let $f_i(y) = 1$ for $y \in G^*$, when $i \neq i_k$. Then, the infinite product

$$f(y) = \prod_{i=1}^{\infty} f_i(y), \quad y \in G^*,$$

is a well-defined positive-definite function on G^* such that, by (2),

$$f(y_{n_k}) = \prod_{s=1}^{k} \gamma^{(i_s)}(y_{n_k}^{(i_s)}).$$

By the above construction $f(y_{n_k})$ does not converge as $k \to \infty$.

THEOREM. *Let G an infinite discrete FC-group. Then every infinite subset $E \subseteq G$ contains an infinite Sidon set.*

PROOF. We may suppose, without loss of generality, E and G countable. Let $\mathfrak{Z}(G)$ be the center of G and $\tilde{G} = G/\mathfrak{Z}(G)$; then $\tilde{G}$ is isomorphic to a subgroup of the weak direct product G^* of countably many finite groups (see e.g. [5, p. 124, Corollary]). Moreover, since G^* is discrete, the restriction of G^* to $\tilde{G}$ induces an isometry of $B(G^*)$ onto $B(\tilde{G})$ [2, 2.31]. Denote by φ: $G \mapsto \tilde{G}$ the canonical projection. If $\varphi(E)$ is finite, then, up to a translation, $\mathfrak{Z}(G)$ contains infinitely many elements of E and the theorem follows. If this is not the case, there are infinitely many distinct elements y_n in $\varphi(E)$. For every n choose $x_n \in E \cap \varphi^{-1}(y_n)$; if (y_n) contains an infinite Sidon set for $B(\tilde{G})$, (x_n) will contain an infinite Sidon set for $B(G)$ (see [2, 2.26]). Let δ_{y_n} be the unit mass concentrated in y_n. By [1, Theorem 1] we have to prove that there is a subsequence of (δ_{y_n}) which is equivalent in $C^*(\tilde{G})$ (i.e. the completion of

$l^1(\tilde{G})$ in the spectral norm) to the usual l^1-basis (see [6, p. 2411] for a definition). By the basic lemma, $(\delta_{y_n} + \delta_{y_n^{-1}})$ (or $(\delta_{y_n} - \delta_{y_n^{-1}})$ does not contain weak Cauchy subsequences; observing that (δ_{y_n}) and $(\delta_{y_n} + \delta_{y_n^{-1}})$ (or $(\delta_{y_n} - \delta_{y_n^{-1}})$) are simultanously equivalent or not to the usual l^1-basis, the theorem is a consequence of the following

LEMMA. *Let (h_n) be a sequence of hermitian elements of a C^*-algebra A and suppose that (h_n) does not contain weak Cauchy subsequences. Then it contains a subsequence equivalent to the usual l^1-basis.*

PROOF. Let A_h be the real Banach space of all hermitian elements of A. Then, as it is easily seen, (h_n) does not contain weak Cauchy subsequences for the weak topology induced on A_h by the hermitian functionals on A. By applying Rosenthal's theorem [6, Main Theorem], we get a subsequence (h'_n) equivalent to the real l^1-basis. Let $c_n = a_n + ib_n$, $n = 1, \ldots, N$, a_n and b_n real numbers and, say, $\sum_{n=1}^N |a_n| \geqslant \sum_{n=1}^N |b_n|$. Let p be a positive linear functional of norm 1 (see [7, 1.5.4]) such that $|p(\sum_{n=1}^N a_n h_n')| \geqslant \frac{1}{2}\|\sum_{n=1}^N a_n h'_n\|$. Then, if $\delta > 0$ is such that $\|\sum_{n=1}^N a_n h'_n\| \geqslant \delta \sum_{n=1}^N |a_n|$ we get

$$\left\|\sum_{n=1}^N c_n h'_n\right\| \geqslant \left|p\left(\sum_{n=1}^N c_n h'_n\right)\right| \geqslant \left|p\left(\sum_{n=1}^N a_n h'_n\right)\right|$$
$$\geqslant \frac{1}{2}\left\|\sum_{n=1}^N a_n h'_n\right\| \geqslant \frac{\delta}{2}\sum_{n=1}^N |a_n| \geqslant \frac{\delta}{4}\sum_{n=1}^N |c_n|.$$

REMARK 1. It was announced in [6] that Rosenthal's theorem has been extended to the complex case by L. Dor. Therefore our lemma, that we reported for completeness, would be a particular case of Dor's result.

REMARK 2. It is worth mentioning that Rosenthal's theorem gives another proof of the fact that every infinite subset of a discrete abelian group contains an infinite Sidon subset. Indeed, given a sequence (x_n) in the abelian discrete group G, there is a character $\gamma \in \Gamma$ such that $\gamma(x_n)$ does not converge. If not, putting $\mu(\gamma) = \lim_{n\to\infty} \gamma(x_n)$, $\mu(\gamma)$ is a measurable multiplicative function on Γ and hence $\mu(\gamma) = \gamma(x)$ for some $x \in G$. On the other hand, for every $f \in L^1(\Gamma)$, $\hat{f}(x) = \lim_{n\to\infty} \hat{f}(x_n) = 0$, which is absurd.

REMARK 3. The main theorem can also be proved, however, without using Rosenthal's theorem: this fact has been pointed out by the referee and independently by the authors after the submission of the paper. The referee's sharper argument is reported below; indeed, it is proved that the set $\{y_{n_k}\}_{k=1}^\infty$, constructed in the proof of the basic lemma, is a Sidon set. To see this, let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence taking the values ± 1. Redefining the characters $\gamma^{(i_k)}$ appropriately, we can arrange matters so that $\varepsilon_k \operatorname{Re} \prod_{s=1}^k \gamma^{(i_s)}(y_{n_k}^{(i_s)}) \geqslant 0$ for all k, and so that this quantity is at least $\frac{1}{2}$ whenever $y_{n_k}^{(i_k)}$ has order greater than 2. Form the function f as in the basic lemma. Then $\varepsilon_k \operatorname{Re} f(y_{n_k}) \geqslant 0$ for all k, and $\varepsilon_k \operatorname{Re} f(y_{n_k}) \geqslant \frac{1}{2}$ if the order of $y_{n_k}^{(i_k)}$ is greater than 2. If none of the $y_{n_k}^{(i_k)}$ have order 2, then $|\operatorname{Re} f(y_{n_k}) - \varepsilon_k| \leqslant \frac{1}{2}$ for all k, and $\{y_{n_k}\}_{k=1}^\infty$ is a Sidon set, by Cygan's Theorem 1. If there are indices k for which $y_{n_k}^{(i_k)}$ has order 2, then, working only with these indices, we can form a second, positive definite, product function g such that g takes the values 0 and $\pm$ 1 only, and such that

$g(y_{n_k}) = \varepsilon_k$ whenever $y_{n_k}^{(i_k)}$ has order 2. Let $u = g/5 + (4/5)\operatorname{Re} f$. Then $|u(y_{n_k}) - \varepsilon_k| \leqslant \frac{4}{5}$ for all k, and $\{y_{n_k}\}_{k=1}^{\infty}$ is a Sidon set.

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