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ON A CONVEXITY PROPERTY OF THE RANGE OF A MAXIMAL MONOTONE OPERATOR

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ABSTRACT. An example is given which shows that the closure of the range of a maximal monotone operator from a (nonreflexive) Banach space into its dual is not necessarily convex.

Introduction. Let X be a real Banach space with dual X^* and let T: $X \to 2^{X^*}$ be a maximal monotone operator with domain $D(T)$ and range $R(T)$. In general $R(T)$ is not a convex set (cf. [4]) but it is known that when X is reflexive, the (norm) closure of $R(T)$ is convex (cf. [5]). Without reflexivity, the convexity of cl $R(T)$ is still true when T is the subdifferential of a lower semicontinuous proper convex function (cf. [1]), or more generally, when the associated monotone operator T_1: $X^{**} \to 2^{X^*}$ is maximal (cf. [2] where the proof is given under a slightly stronger assumption). Here T_1 denotes the operator whose graph is defined by

$$\operatorname{gr} T_1 = \{(x^{**}, x^*) \in X^{**} \times X^*; \exists \text{ a net } (x_i, x_i^*) \in \operatorname{gr} T \text{ with } x_i \text{ bounded}, x_i \to x^{**} \text{ weak}^{**} \text{ and } x_i^* \to x^* \text{ in norm}\}.$$

(X is identified as usual to a subspace of its bidual X^{**}.) The question was raised some years ago as to whether or not the convexity of cl $R(T)$ holds in general.

In this note we answer this question negatively. We exhibit a (everywhere defined and coercive) maximal monotone operator from l^1 to 2^{l^∞} whose range has not a convex closure. Our construction is based on a result of [3].

Example. Let A: $l^1 \to l^\infty$ be the bounded linear operator defined by

$$(Ax)_n = \sum_{m=1}^{\infty} x_m \alpha_{mn}$$

for $x = (x_1, x_2, \ldots) \in l^1$, where $\alpha_{mn} = 0$ if $m = n$, $\alpha_{mn} = -1$ if $n > m$ and $\alpha_{mn} = +1$ if $n < m$. Let J: $l^1 \to 2^{l^\infty}$ be the usual duality mapping:

$$Jx = \{|x|_{l^1}(s(x_1), s(x_2), \ldots)\}$$

where s: $\mathbf{R} \to 2^{\mathbf{R}}$ is given by $s(t) = -1$ if $t < 0$, $s(t) = [-1, +1]$ if $t = 0$ and $s(t) = +1$ if $t > 0$. For $\lambda > 0$, the mapping $\lambda J + A$ is clearly max-

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imal monotone and coercive. It was shown in [3] that there exists $\lambda > 0$ such that $R(\lambda J + A)$ is not dense in l^∞ (for the norm topology of l^∞).

PROPOSITION. *Let $\lambda > 0$ be such that $R(\lambda J + A)$ is not dense in l^∞. Then* cl $R(\lambda J + A)$ *is not a convex set.*

PROOF. Assume by contradiction that cl $R(\lambda J + A)$ is convex. Since $y \in$ cl $R(\lambda J + A)$ implies that the whole line $\{ry; r \in \mathbf{R}\}$ is contained in cl $R(\lambda J + A)$, cl $R(\lambda J + A)$ would be a proper closed subspace of l^∞. Consequently there would exist a nonzero $\mu \in (l^\infty)^*$ which vanishes on $R(\lambda J + A)$. We will show that this is impossible.

Let $\mu \in (l^\infty)^*$ vanish on $R(\lambda J + A)$. Denoting by $\beta\mathbf{N}$ the Stone-Čech compactification of $\mathbf{N}$, one can identify l^∞ to the space $C(\beta\mathbf{N})$ of the continuous real-valued functions on $\beta\mathbf{N}$ and $(l^\infty)^*$ to the space $\mathfrak{M}(\beta\mathbf{N})$ of the Radon measures on $\beta\mathbf{N}$. We first show that $\mu_i = 0$ for each $i \in \mathbf{N}$, where μ_i denotes the μ-measure of $\{i\} \subset \beta\mathbf{N}$. The points

$$y_t = (\lambda t, \lambda - 1/2, -\lambda - 1/2, 0, 0, \dots),$$

$t \in [-1, +1]$, belong to the image of $(0, 1/2, -1/2, 0, 0, \dots)$ by $(\lambda J + A)$. Thus $\langle \mu, y_t \rangle = 0$ for all $t \in [-1, +1]$, which implies $\mu_1 = 0$. Considering the image of $(0, 0, 1/2, -1/2, 0, 0, \dots)$ by $(\lambda J + A)$, one similarly gets $\mu_2 = 0$. And so on. We now prove that $\mu = 0$, i.e. that $\langle \mu, y \rangle = 0$ for all $y \in l^\infty$. Let $y = (y_1, y_2, \dots) \in l^\infty$. The image of $(k, -k, 0, 0, \dots)$, $k > 0$, by $(\lambda J + A)$ is

$$\{(2\lambda k - k, -2\lambda k - k, 2\lambda ks, 2\lambda kt, \dots); s, t, \dots \in [-1, +1]\};$$

thus, if k is chosen sufficiently large, this image contains the point

$$\tilde{y} = (2\lambda k - k, -2\lambda k - k, y_3, y_4, \dots).$$

But $\tilde{y}_j = y_j$ for almost every $j \in \mathbf{N}$. Since $\mu_i = 0$ for $i \in \mathbf{N}$, it follows that $\langle \mu, y \rangle = \langle \mu, \tilde{y} \rangle$, which is zero since $\tilde{y} \in R(\lambda J + A)$. Q.E.D.

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