IRREDUCIBLES IN THE LANDWEBER NOVIKOV ALGEBRA

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ABSTRACT. All the irreducible and reducible elements in the Landweber Novikov algebra are determined. A full set of relations mod reducibles is given.

1. Introduction. Let $S^*$ denote the Landweber Novikov algebra, and let $\bar{S}$ be the kernel of the augmentation map. The aim of this paper is to compute $Q(S^*) = \bar{S}/\bar{S}^2$, the module of irreducibles.

For every exponent sequence $\alpha$ with only finitely many nonzero terms, Landweber [1] and Novikov [2] define an operation $s_\alpha \in S^*$. Moreover, the $s_\alpha$'s form a basis for $S^*$ as a $\mathbb{Z}$-module.

For every exponent sequence $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$, let $||\alpha|| = \sum i\alpha_i$, and $|\alpha| = \sum \alpha_i$. Let $\Delta(\alpha)$ denote the exponent sequence all of whose elements are zero except 1 in the $\alpha$th place. Our main theorem is

**Theorem 1.1.**

(a) $Q(S^*)$ is generated by \{ $s_p^{s^*\Delta(1)}, s_p^{s^*\Delta(2)}$ | $p$ prime, $n > 0$ \}, with the only relations $p^{s_p^{s^*\Delta(1)}} \in \bar{S}^2$ for $n > 2$ and every $p$, $p^{s_p^{s^*\Delta(1)}} \in \bar{S}^2$ for $p \neq 2$, $p^{s_p^{s^*\Delta(2)}} \in \bar{S}^2$ for $n > 1$ and $2(s^{s^*\Delta(2)} + s_2^{s^*\Delta(1)}) \in \bar{S}^2$.

(b) All the $s_\alpha$'s are reducible except for $\alpha = p^n\Delta(1), p^n\Delta(2), 2p^n\Delta(1)$. The only relations between irreducibles are those specified in (a) and $s^{s^*\Delta(1)} + s_2^{s^*\Delta(1)} \in \bar{S}^2$ for $p \neq 2$ and $n > 0$.

Our main computational tool is the following theorem due to Landweber [1].

Let $S_*^*$ be the dual algebra to $S^*$. Let $s_\alpha$ be the dual basis to $s_\alpha$. Then $S_*^*$ is a polynomial algebra with generators \{ $s_{\Delta(\alpha)}(a)$ | $a \geq 1$ \} and

**Theorem 1.2.** The diagonal in $S_*^*$ is given by

$$\phi_\alpha(s_{\Delta(\alpha)}) = \sum_{||\alpha||+i = a} \binom{i+1}{\alpha} s_{\alpha} \otimes s_{\Delta(i)}.$$

**ADDED IN PROOF.** While writing this paper I heard that Aikawa [3] got the same results. I would like to thank Shibata for reading this paper and correcting many of the mistakes appearing in the original version.

2. Definition. If $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$, let $v_p(\alpha) = \min_i \{ v_p(\alpha_i) \}$ and $v_p(\alpha) = \max \{ 0, v_p(\alpha) - v_p(n) \}$.

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Let us say that two exponent sequences \( \alpha = (a_1, \ldots, a_n, \ldots) \), \( \beta = (b_1, \ldots, b_n, \ldots) \) are disjoint if \( a_i b_i = 0 \) for every \( i \).

**Theorem 2.1.** (a) For every \( \alpha, \beta \) we have \( s_\alpha \circ s_\beta = \lambda s_{\alpha + \beta} + \sum \lambda_i s_\alpha \) where \( |\alpha_i| < |\alpha + \beta| \). If \( \alpha \) and \( \beta \) are disjoint, then \( \lambda = 1 \). Moreover, \( v_p(\lambda_i s_\alpha) \leq \min\{v_p(\alpha), v_p(\beta)\} \).

(b) For \( n > 1 \) there exists a \( \lambda \in \mathbb{Z} \) such that \( \lambda s_{n\Delta(a)} \equiv \sum \lambda_i s_\alpha \mod S^2 \), where \( |\alpha| < n \) and \( v_q(\lambda_i s_\alpha) \leq v_q(n) \) for every prime \( q \). Moreover, \( \lambda = 1 \) if \( n \) is not a power of a prime, and \( \lambda = p \) if \( n = p^k \) for some prime \( p \).

**Proof of (a).** We will prove (a) by passing to the dual. That is, if \( \phi_*(\gamma) = \lambda \sigma_\alpha \otimes \sigma_\beta + \cdots \) with \( \lambda \neq 0 \), then \( |\gamma| < |\alpha + \beta| \) unless \( \gamma = \alpha + \beta \). This will follow from 1.2 by trivial induction on \( |\gamma| \). We also have to show that \( \min\{v_p(\alpha), v_p(\beta)\} \geq v_p(\gamma) - v_p(\beta) \). Let \( r = v_p(\gamma) \), i.e. \( \gamma = p^r \beta \). Hence,

\[
\phi_*(\gamma) = \phi_*(\sigma_\beta)^{p^r} = \left( \sum \mu_i \sigma_\alpha \otimes \sigma_\beta \right)^{p^r},
\]

and we will get the results from the following lemma.

**Lemma 2.2.** If \( \sum \gamma_i z_i = \sum \lambda_i z_i \), where the \( z_i \)'s are monomials in the \( y_i \)'s, then \( v_p(z_i) + v_p(y_i) \geq r \), where \( v_p(z) = \max\{r \mid \exists y \text{ with } z = y^{p^r}\} \).

**Proof of 2.1 (b).** From (a) we have that if \( k + l = n \), then

\[
s_{k\Delta(a)} \circ s_{l\Delta(a)} = \binom{n}{k} s_{n\Delta(a)} + \sum \lambda_i s_\alpha,
\]

where \( |\alpha| < n \) and \( v_p(\lambda_i s_\alpha) \leq v_p(n) \) for every prime \( p \). But g.c.d. \( \left( \binom{n}{k} \right) \) is the same \( \lambda \) defined in the theorem, and, hence, we can take an appropriate linear combination of the above relations to get (b).

**Corollary 2.3.** For every \( n \) and \( \alpha \) we have that

\[
n_s = \sum_{p, \alpha, i} \lambda_{p, \alpha, i} s_{p\Delta(a)} \mod S^2
\]

where \( i \leq v_p(n_s) \).

**Proof.** The proof is by induction on \( |\alpha| \). If \( \alpha \) is not of the form \( m\Delta(a) \), then there are disjoint \( \beta, \gamma \) such that \( \alpha = \beta + \gamma \). Then by 2.1(a), \( n_{s_\alpha} \equiv \sum N_{\lambda_i s_\alpha} \) with \( v_p(n_{\lambda_i s_\alpha}) \leq v_p(n_{s_\alpha}) \) and \( |\alpha| < |\alpha| \). Apply now the induction hypothesis to \( \alpha_i \) and \( n_{\lambda_i} \).

If \( \alpha = m\Delta(a) \), do the same using 2.1(b).

**Lemma 2.4.** For every \( a \neq b \),

(a) \( s_{\Delta(a)} \circ s_{\Delta(b)} = s_{\Delta(a) + \Delta(b)} + (b + 1)s_{\Delta(a + b)} \).

(b) \( s_{2\Delta(a)} \circ s_{2\Delta(b)} = \lambda s_{\Delta(a) + 2\Delta(b)} + (b + 1)s_{\Delta(a) + \Delta(a + b)} \).

(c) \( s_{2\Delta(b)} \circ s_{\Delta(a)} = \lambda s_{\Delta(a) + 2\Delta(b)} + (a + 1)s_{\Delta(b) + \Delta(a + b)} + \left( \frac{a + 1}{2} \right) s_{\Delta(a + 2b)} \).

\( \lambda \) is the same as in (b) and \( \lambda = 1 \) if \( a \neq b \).

(d) \( s_{2\Delta(a)} \equiv 0 \mod S^2 \) for every \( a \neq 1, 2 \).

**Proof.** (a), (b), (c) are routine computations. To prove (d) we will have to separate cases.
(1) \( a \) odd, \( a \neq 1 \). Write \( a = b + c \) with \( b - c = 1 \). Then by (a), \( [s_{\Delta(b)}, s_{\Delta(c)}] = (b - c)s_{\Delta(a)} = s_{\Delta(a)} \).

(2) \( a \) even, \( a \neq 2 \). Write \( a = b + c \) with \( b - c = 2 \). Then as in (1) we get \( 2s_{\Delta(a)} \equiv 0 \mod S^2 \).

Let \( a = 2 + 2b \). Using (b), (c) and (a) we get that
\[
[s_{\Delta(2)}, s_{2\Delta(b)}] = (b - 2)s_{\Delta(b) + \Delta(b + 2)} + 3s_{\Delta(a)}
\]
and
\[
s_{\Delta(b + 2)} \circ s_{\Delta(b)} = s_{\Delta(b) + \Delta(b + 2)} + (b + 1)s_{\Delta(a)}.
\]
Combining both we get
\[
s_{\Delta(a)} \equiv [(b + 1)(b - 2) - 2]s_{\Delta(a)} \mod S^2.
\]
But \((b + 1) \cdot (b - 2)\) is even and, hence, \( s_{\Delta(a)} \in S^2 \).

**Lemma 2.5.**

\[
s_{p^*\Delta(a)} \circ s_{p^*\Delta(b)} = (b + 1)P^*s_{p^*\Delta(a) + \Delta(b)} + s_{p^*\Delta(a) + \Delta(b)}
\]
\[\quad + \sum_{c; k < n} \lambda_{c,k}s_{p^*\Delta(c)} \mod S^2 \text{ for } a \neq b.
\]
\[
s_{p^*\Delta(a)} \circ s_{2p^*\Delta(b)} = \lambda^2 s_{p^*\Delta(a) + 2\Delta(b)} + (b + 1)P^*s_{p^*\Delta(a) + \Delta(b + a + b)}
\]
\[\quad + \sum_{c; k < n} \lambda_{c,k}s_{p^*\Delta(c)} \mod S^2.
\]
\[
s_{2p^*\Delta(b)} \circ s_{p^*\Delta(a)} = \lambda^2 s_{p^*\Delta(a) + 2\Delta(b)} + (a + 1)P^*s_{p^*\Delta(a) + \Delta(b + a) + b)}
\]
\[\quad + \left(\frac{a + 1}{2}\right)P^*s_{p^*\Delta(a + 2b)} + \sum_{c; k < n} \lambda_{c,k}s_{p^*\Delta(c)} \mod S^2.
\]

The constant \( \lambda \) appearing in (b) and (c) is the same \( \lambda \) as in 2.4.

(d) For every \( a \neq 1, 2 \) we have \( s_{p^*\Delta(a)} \in S^2 \).

**Proof.** The proof of (a), (b) and (c) are identical, so we will prove (b). By 2.1 we have that
\[
s_{p^*\Delta(a)} \circ s_{2p^*\Delta(b)} = \lambda^2 s_{p^*\Delta(a) + 2\Delta(b)} + \sum \lambda_i s_{\alpha_i}
\]
with \( |\alpha_i| < 3p^n \), and that for every prime \( q \neq p \), \( \nu_q(\lambda_i s_{\alpha_i}) = 0 \). We want to show that the only possible \( \alpha_i \) in the sum with \( \nu_p(\alpha_i) \geq n \) is \( p^n(\Delta(b) + \Delta(a + b)) \). This will imply (b) by Corollary 2.3.

But if \( \alpha_i = p^n\beta \) and \( |\alpha_i| < 2p^n \), then \( \beta \) must be of the form \( \Delta(t) + \Delta(s) \) or \( 2\Delta(t) \). An immediate check leaves the only possibility \( \beta = \Delta(a) + \Delta(a + b) \).

(d) The proof is by induction on \( n \), the case \( n = 0 \) having been done in 2.4(d). For \( n > 0 \) one follows the proof of 2.4(d). The only extra fact which is needed is that \( ps_{p^*\Delta(a)} \in S^2 \) for \( a > 2 \), but this will follow from \( ps_{p^*\Delta(a)} \equiv \sum_{t < n; \beta} \lambda_{t; b} s_{p^*\Delta(b)} \) and our induction hypothesis. (Note that in the above sum we have \( b > 2 \), so induction applies.)

**Proof of Theorem 1.1.** Let \( ||\alpha|| = n \) with \( n \) not of the form \( p^m \) or \( 2p^m \).

Then \( s_{\alpha} \equiv \sum_{t; \beta, i} \lambda_{i; \beta} s_{p^*\Delta(a)} \mod S^2 \) where \( a \neq 1, 2 \). Then by 2.5(d), \( s_{\alpha} \) is reducible.
If \( \|\alpha\| = p^n \text{ or } 2p^n \), but \( \alpha \neq p^n\Delta(1), p^n\Delta(2) \text{ or } 2p^n\Delta(1) \), then \( \nu_p(\alpha) < n \) and 
\[ s_a = \sum_i \lambda_i a^{s_p(\Delta(a))} \text{ with } a \neq 1, 2 \text{ and, as before, } s_a \text{ is reducible.} \]

As for the remaining cases, we have already shown in the proof of 2.5(d) that \( p^s_{p^*\Delta(a)} \in \tilde{S}^2 \) for \( a > 2 \). The same proof works if \( a = 1 \) and \( p > 2 \). To show that these are the only relations, look at:

\[ \phi_s(\sigma_{p^*\Delta(1)}) = (\sigma_{\Delta(1)} \otimes 1 + 1 \otimes \sigma_{\Delta(1)})^p. \]

So in any relation where \( s_{p^*\Delta(1)} \) appears, it is with coefficient divisible by \( p \). Hence, it is irreducible.

\[ \phi_s(\sigma_{p^*\Delta(2)}) = (\sigma_2 \otimes 1 + 2\sigma_1 \otimes \sigma_1 + 1 \otimes \sigma_2)^p. \]

Hence, the only relation in which \( s_{p^*\Delta(2)} \) appears with a coefficient which is not divisible by \( p \) is

\[ s_{p^*\Delta(1)} \circ s_{p^*\Delta(1)} = 2p^n s_{p^*\Delta(2)} + \left( \frac{2p^n}{p^n} \right) s_{2p^*\Delta(1)} + \cdots. \]

Similarly, the previous relation is the only interesting one for \( s_{2p^*\Delta(1)} \). The other terms in this expression are all in \( \tilde{S}^2 \).

If \( p > 2 \) we have

\[ \left( \frac{2p^n}{p^n} \right) \equiv 2p^n \equiv 2 \mod p. \]

Hence, \( 2(s_{2p^*\Delta(1)} + s_{2p^*\Delta(2)}) \in \tilde{S}^2 \). We also have \( p(s_{2p^*\Delta(1)} + s_{p^*\Delta(2)}) \in \tilde{S}^2 \) and we get our theorem for \( p \neq 2 \). If \( p = 2 \) we have

\[ \left( \frac{2p^n}{p^n} \right) \equiv 2 \mod 4. \]

Hence, \( 2s_{2^{n+1}\Delta(1)} \in \tilde{S}^2 \) for every \( n > 1 \), which finishes our proof.

**References**


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