ON PROPERLY EMBEDDING PLANES IN 3-MANIFOLDS

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ABSTRACT. In this paper we prove an analog of the loop theorem for a certain class of noncompact 3-manifolds. In particular, we show that the existence of a "nontrivial" proper map of a plane into a 3-manifold implies the existence of a nontrivial proper embedding of a plane into a 3-manifold.

Introduction. In this paper we prove an analog of the loop theorem for a certain class of noncompact 3-manifolds. More precisely we show that the existence of a "nontrivial" proper map of a plane into a noncompact eventually end-irreducible 3-manifold implies the existence of a "nontrivial" proper embedding of a plane into that 3-manifold. We remark that an eventually end-irreducible 3-manifold is essentially a 3-manifold which has an infinite hierarchy. A discussion of proper homotopy and related topics is given in [1].

Notation. All spaces are simplicial complexes and all maps are piecewise linear. We use the notation of Brown and Tucker [1] without change. A 3-manifold is eventually end-irreducible at the end [a] if there is an exhausting sequence \(\{M_n\}\) of compact 3-dimensional submanifolds of \(M\) and a compact subset \(C \subset \text{int}(M_1)\) with the following property. (a) If \(A\) is the component of \(M - M_n\) determined by the end [a] and if \(F\) is a component of \(\text{Fr}(A)\), then the inclusion map \(\pi_1(F) \to \pi_1(M - C)\) induces a monomorphism.

Results. The main result in this paper is the following

Theorem. Let \(M\) be a 3-manifold and let \([a]\) be an end of \(M\) which is eventually end-irreducible. Let \(f: (R^2, \ast) \to (M, [a])\) be a proper map which carries the unique end \([\ast]\) of \(R^2\) to the end \([a]\) of \(M\). Assume \(\pi_1(f)\) is nontrivial. Then there is a proper embedding \(g: (R^2, \ast) \to (M, [a])\) such that \(\pi_1(g)\) is nontrivial.

Proof. We rely heavily on the proof of the loop theorem in [3]. Let \(C\) and \(\{M_n\}\) be as in the definition of eventually end-irreducible at the end \([a]\). Since \(\pi_1(f)\) is nontrivial, we may assume that there is a disk \(D_0 \subset R^2\) so that \(f^{-1}(C) \subset D_0\) and if \(\lambda\) is an essential loop in \(R^2 - D_0\), then \(f(\lambda)\) is essential in \(M - C\).

By a subsequence of the \(M_n\)'s we may assume that \(D_0 \subset f^{-1}(M_1)\) and that if \(f(R^2)\) meets a component \(A\) of \(M - M_n\), then \(A\) is determined by \([a]\). Next,
by an arbitrarily small proper homotopy of $f$, we assume that $f(R^2) \cap \partial M = \emptyset$; that $f$ is in general position with respect to $\text{Fr}(M_n)$ for every $n$, and that singularities of $f$ consist of double curves, triple points and nodes. It follows that components of $f^{-1}(M_n)$ are compact submanifolds of $R^2$ (disks with holes) one of which, say $\mathcal{D}_n$, contains $\mathcal{D}_0$.

We claim that without changing $\pi_1(f)$ we can change $f$ to a new map with all the above properties and so that each $\mathcal{D}_n$ is a disk. (If $\pi_2(a)$ is trivial, the change may be accomplished by a proper homotopy.) To see this consider first $\mathcal{D}_1$. Exactly one of its boundary components (which we call the outer boundary) bounds a disk in $R^2$ containing $\mathcal{D}_1$. Any inner boundary component of $\mathcal{D}_1$ bounds a disk $\mathcal{D}$ in $R^2$ with $f(\mathcal{D}) \subset (M - C)$ since $f^{-1}(C) \subset \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_n$. Since $f(\partial \mathcal{D}) \subset \text{Fr}(M_1)$, property (a) above implies that we can redefine $f$ on $\mathcal{D}$ so that $f(\mathcal{D}) \subset \text{Fr}(M_1)$. We may restore general position by a small proper homotopy. Assume then that $\mathcal{D}_1$ is a disk.

Next consider $\mathcal{D}_n$ for $n > 1$. If $\mathcal{D}_n$ is not a disk, we can find an inner boundary component of $\mathcal{D}_n$ which bounds a disk $\mathcal{D}$ such that $f(\mathcal{D}) \subset M_n - M_{n-1}$ and so that $f$ still has the general position properties mentioned above. Notice that the above change does not increase the number of boundary components of $\mathcal{D}_j$ for all $j$ and that $f|\mathcal{D}_{n-1}$ is unaffected. Thus by induction we may redefine $f$ so that all of the $\mathcal{D}_n$'s are disks and it should be clear that our new $f$ is still a proper map. Finally since we have not changed $f$ on the outer boundaries of the $\mathcal{D}_n$'s we have not changed $\pi_1(f)$.

We assume now that $\mathcal{D}_n$ is a disk for each $n$, and we note that $f|\partial \mathcal{D}_n$ is essential in $M - C$. If $F_n$ is the component of $\text{Fr}(M_n)$ containing $f(\partial \mathcal{D}_n)$, then according to the loop theorem [3], there is an embedded disk $E_n$ in $M_n$ with $E_n \cap \partial M_n = E_n \cap F_n = \partial E_n$. Moreover $\partial E_n$ is not in $\ker(\pi_1(F_n) \to \pi_1(M - C))$.

Recall that in the proof of the loop theorem [3] the disk $E_n$ is constructed as follows: One constructs a tower of 2-sheeted coverings of regular neighborhoods of $f(\mathcal{D}_n)$. At the top of the tower a disk is selected in the boundary of the regular neighborhood and then brought down the tower by cuts. We observe that $E_n$ will be in general position with respect to $\text{Fr}(M_j)$ for $j < n$ since $f|\mathcal{D}_n$ is in general position with respect to that surface. Thus we can assume that $E_n \cap \text{Fr}(M_j)$ is a collection of disjoint simple loops. We claim that the possibilities for $E_n \cap \text{Fr}(M_j)$, $j < n$, are essentially determined by the loops $ff^{-1}\text{Fr}(M_j)$ and that there are only finitely many possibilities given $ff^{-1}\text{Fr}(M_j)$. This can be seen by observing that $E_n \cap \text{Fr}(M_j)$ is a collection of loops which are essentially composed of arcs in

$$L = \left\{x \in f^{-1}(M_j) : \{x\} \neq f^{-1}f(x)\right\}$$

where no arc in $L$ can be used more than twice. (Alternatively, see the addendum to Theorem III.5 in [2].)

We assert next that for each $n$ we can choose an $n$-tuple $(l_{n,1}, l_{n,2}, \ldots, l_{n,n})$ of simple loops, concentric in the given order on $E_n$ so that $l_{n,i}$ is a component of $E_n \cap \text{Fr}(M_i)$, and so that each $l_{n,i}$ is essential in $M - C$. Clearly we must choose $l_{n,n} = \partial E_n$. Some component of $E_n \cap \text{Fr}(M_{n-1})$ is essential in $M - C$.
since $l_{n,n}$ is essential. Choose one and call it $l_{n,n-1}$. Now $l_{n,n-1}$ bounds a subdisk of $E_n$ and some component of the intersection of this subdisk with $\text{Fr}(M_{n-1})$ is essential in $M - C$ since $l_{n,n-1}$ is essential. The truth of the assertion then is demonstrated by a finite induction from the top down.

In the choice of the $l_{n,i}$ we have a certain amount of freedom. Let us pick $l_{n,i}$ on $\text{Fr}(M_i)$ as an innermost loop on $E_n$ which is essential in $M - C$. If $A_{n,i}$ is the subannulus of $E_n$ bounded by $l_{n,i}$ and $l_{n,i+1}$, then $A_{n,i}$ meets $\partial M_{i+1}$ in $l_{n,i+1}$ together with loops which are inessential in $M - C$. These last loops are also inessential on $\partial M_{i+1}$ and hence bound disks there.

It follows that we can define $E_n$ such that $E_n \cap \text{Fr}(M_j) \subset E_n \cap \text{Fr}(M_j)$ and the disk bounded by $l_{n,j}$ lies within $M_j$ for all $j < n$.

As noted above there are only a finite number of distinct terms in the sequence $\{l_{n,j}\}$. Let $l_1 = l_{n,1}$ for an infinite subsequence $\{n_i\}$ of $\{n\}$. Then choose $l_2 = l_{n,2}$ for an infinite subsequence $\{n_i\}$ of $\{n_j\}$. By induction we construct a sequence of simple loops $\{l_k\}$ and a sequence of integers $n_k$ so that for $1 < j < k$, $l_j = l_{n,j}$. It follows that for each positive integer $m$, the pair $l_m, l_{m+1}$ bounds the annulus $A_{n,m}$ on $E_n$ whenever $k > m + 1$. Among these annuli, let $A_m$ be one which misses as many of the $A_{n,i}$'s as possible. Let $A_0$ be the subdisk of $E_n$ bounded by $l_1$.

Now $\bigcup_{m=0}^{\infty} g(A_m)$ is a singular plane in $M$, which contains the loops $\{l_k\}$ as a concentric proper sequence. Let $g: R^2 \to M$ be a map which carries $\{x: ||x|| < m + 1\}$ homeomorphically onto $A_m$.

We assert that $g$ is a proper map. The only way this can fail is if for some $k$ we have $A_m \cap M_k \neq \emptyset$ for an infinite number of integers $m$. But if $m$ is one such integer, this means that the annulus on $E_n$ bounded by $l_m$ and $l_{m+1}$ meets $\text{Fr}(M_k)$ in some collection of loops for every $i \geq m + 1$ (recall the choice of $A_m$). But then for $i$ large the number of disjoint simple loops in $E_n \cap \text{Fr}(M_k)$ is as large as we wish. But we pointed out above that this number was bounded by a number independent of $n$. It follows that $g$ is a proper map.

Observe that the loop $l_j$ bounds a subdisk $A_j$ of $E_{n_k}$ for $j < k$ such that $A_j \subset M_j$ and $l_i \subset A_j$ for $i < j$. Since $E_{n_k}$ is an embedded disk it follows that $A_m \cap l_j$ is empty if $i \neq m$ or $m + 1$. We will now describe how to make a sequence of cuts so that we obtain a proper embedding of a plane in $M$ from our map $g$.

We let $\text{Fr}_{n}$ be $\{x: ||x|| < n\}$ and $A_n$ be the closure of $\text{Fr}_{n+1} - \text{Fr}_{n}$. Since $g|\text{Fr}_{1}$ and $g|A_1$ are embeddings the map $g|\text{Fr}_{2}$ has no branch points or triple points, furthermore since $g|\text{Fr}_{1} \subset M_1$, we know that the singular set of $g|\text{Fr}_{2}$ does not approach the boundary of $\text{Fr}_{2}$. Since $l_1 \subset g(\text{Fr}_{1})$, by cutting and pasting we can obtain a new map $h: \text{Fr}_{2} \to M_2$ such that

1. $h|\text{Fr}_{1} = g|\text{Fr}_{1}$,
2. $h(\partial \text{Fr}_{2}) = l_2$,
3. $h^{-1}\text{Fr}(M_1) \subset g^{-1}\text{Fr}(M_1)$.

We assume that $h$ has been defined on $\text{Fr}_{n}$ and proceed inductively by extending $h$ to $\text{Fr}_{n+1}$. We observe first that $g(A_n)$ is an embedding as is $h|\text{Fr}_{n}$ and that $l_1, l_2, \ldots, l_n$ are not in $g(A_n)$. We may define a singular map $h_1$ onto $\text{Fr}_{n+1}$ by

1. $h_1|\text{Fr}_{n} = h$,
2. $h_1|A_n = g|A_n$.
Now the singular set of $h_1$ is made up of a collection of simple double loops none of which meets $\bigcup_{n+1}^{n+1} l_i$. Thus after a sequence of cuts we can use $h_1$ to define $h_2$ on $\bar{\partial}_n$, so that

1. $h_2|_{\bar{\partial}_n} = h_1|_{\bar{\partial}_n}$,
2. $h_2^{-1}(\text{Fr}(M_n)) \subset g^{-1}(\text{Fr}(M_n))$,
3. $h_2(\bar{\partial}_n + 1) = l_{n+1}$.

We now extend $h$ to $\bar{\partial}_{n+1}$ by requiring that $h|_{\bar{\partial}_{n+1}} = h_2|_{\bar{\partial}_{n+1}}$. It follows that we may assume that $h$ has been extended to $R^2$. Now $h$ is proper since $h^{-1}(\text{Fr}(M_n))$ has no more components; then $g^{-1}(\text{Fr}(M_n))$ and $h\{x| \|x\| = n\} = l_n$. Since $h\{x| \|x\| = n\} = l_n$, $\pi_1(h)$ is nontrivial and the theorem follows.

**Bibliography**


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