EMBEDDINGS OF COMPACTA WITH SHAPE DIMENSION IN THE TRIVIAL RANGE

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Abstract. In this paper a loop condition is defined which generalizes the cellularity criterion and applies to compacta with nontrivial shape. It is shown that if \( X, Y \subseteq E^n, n > 5 \), are compacta which satisfy this loop condition and whose shape classes include a space having dimension in the trivial range with respect to \( n \), then \( \text{Sh}(X) = \text{Sh}(Y) \) is equivalent to \( E^n - X \cong E^n - Y \). An application is given to compacta with the shape of a compact connected abelian topological group.

1. Introduction and statements of main results. Recently several individuals have studied special cases of the following general problem: if \( X \) and \( Y \) are compacta in Euclidean \( n \)-space \( E^n \), under what conditions is \( E^n - X \cong E^n - Y \) equivalent to \( \text{Sh}(X) = \text{Sh}(Y) \)? The results of this paper concern compacta whose shape classes include a space having dimension in the trivial range with respect to \( n \). We give a global homotopy condition under which the equivalence holds for such compacta. Before stating our main result we make some definitions.

Definition. Let \( A^n \) be a compact subset of the manifold \( M \). \( X \) is said to satisfy the inessential loops condition (ILC) if for every neighborhood \( U \) of \( X \) in \( M \) there exists a neighborhood \( V \) of \( A \) in \( U \) such that each loop in \( V - X \) which is null-homotopic in \( V \) is also null-homotopic in \( U - X \). (See §2 for the definitions of other loop conditions and a discussion of some of the relations among them.) For any compactum \( X \), the shape dimension of \( X \) (\( \text{Sd}(X) \)) is defined by \( \text{Sd}(X) = \min\{\dim Y : \text{Sh}(X) = \text{Sh}(Y)\} \). We say that \( k \) is in the trivial range with respect to \( n \) if \( 2k + 2 < n \).

Theorem 1. Let \( X \) and \( Y \) be compacta in \( E^n, n > 5 \), satisfying ILC and having shape dimension in the trivial range with respect to \( n \). Then \( E^n - X \cong E^n - Y \) if and only if \( \text{Sh}(X) = \text{Sh}(Y) \).

As a consequence of Theorem 1 we prove the following theorem about compacta with the shape of a topological group. For example, \( A \) and \( B \) in Theorem 2 could be solenoids. Recall that every finite dimensional compact connected abelian topological group is metrizable [16].

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Theorem 2. Let $X, Y \subseteq E^n$, $n \geq 5$, be globally 1-alg compacta and let $A, B$ be compact connected abelian topological groups with $2 \dim A + 2 \leq n$. If $\text{Sh}(X) = \text{Sh}(A)$ and $\text{Sh}(Y) = \text{Sh}(B)$ then the following are equivalent:

(i) $E^n - X \approx E^n - Y$,

(ii) $\text{Sh}(X) = \text{Sh}(Y)$, and

(iii) $A$ and $B$ are topologically isomorphic.

Theorem 1 is related to several other recent results. Chapman [3] proved that if $\dim X, \dim Y < k$ and $3k + 3 < n$, then there are copies $X'$ and $Y'$ of $X$ and $Y$, respectively, in $E^n$ so that $\text{Sh}(X) = \text{Sh}(Y)$ if and only if $E^n - X' \approx E^n - Y'$. Geoghegan and Summerhill [5] refined Chapman's theorem by reducing the unnecessary condition $3k + 3 < n$ to the trivial range and by making more explicit which copies of $X$ and $Y$ are acceptable. Specifically, they required that the copies of $X$ and $Y$ be 1-ULC. Hollingsworth and Rushing [7] improved this result by replacing 1-ULC (which is a local condition) with the small loops condition (which is global). The global condition is more desirable for a weak flatness theorem of this type—see [7] for more details.

The work of Hollingsworth and Rushing is generalized in Theorem 1 since the same conclusion is drawn for compacta which themselves do not necessarily have dimension in the trivial range but merely have the shape of such. Coram, Daverman and Duvall [4] have previously proved Theorem 1 in the special case that $\dim X < n - 3$ and $Y$ is a finite polyhedron with dimension in the trivial range.

Theorem 2 answers a question raised by J. Keesling.

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2. Definitions and notation. Let $X$ be a compactum in the manifold $M^n$. $X$ is said to satisfy the cellularity criterion (small loops condition) if given a neighborhood $U$ of $X$ there exists a neighborhood $V$ of $X$ in $U$ (and a number $\varepsilon > 0$) such that any loop in $V - X$ (any $\varepsilon$-loop in $V - X$) is null-homotopic in $U - X$. $X$ is said to be globally 1-alg in $M$ if given a neighborhood $U$ of $X$ there exists a neighborhood $V$ of $X$ in $U$ such that any loop in $V - X$ which is null-homologous in $U - X$ is null-homotopic in $U - X$.

These loop conditions are closely related. For example if $\dim X < n - 2$, then ILC is equivalent to the small loops condition. On the other hand, if $X$ has the shape of a point, then ILC is equivalent to the cellularity criterion. In case $\text{Sd}(X) < n - 3$, Alexander duality shows that the inclusion induced homomorphism $H_1(V - X) \to H_1(V)$ is an isomorphism. Hence if $\text{Sd}(X) < n - 3$, $X$ globally 1-alg implies $X$ satisfies ILC. Finally, it can easily be seen that if $X$ has the shape of the inverse limit of a sequence of ANR's where each of these ANR's has abelian fundamental group, then $X$ is globally 1-alg whenever $X$ satisfies ILC.

Throughout this paper the symbols $\approx$ and $\simeq$ will have the following meanings: $\approx$ means "is homeomorphic to," "is isomorphic to," or "is topologically isomorphic to," depending on the context; while $\simeq$ means "is homotopic to." $H_\ast$ denotes reduced singular homology, $H^\ast$ Čech cohomology and $H^\ast_\ast$ Alexander cohomology with compact supports [13] all with integer coefficients.
All spaces are assumed to be metric.

For definitions of concepts related to shape theory, the reader is referred to [1] and [10]. For all other definitions consult [12].

3. **Compacta in standard position.** We begin by making a definition [3] which is basic to the entire proof of Theorem 1.

**Definition.** Let $X \subset E^n$ be a compactum and $k = \text{Sd}(X)$. $X$ is in **standard position** if there exist sequences $(P_i)_{i=1}^\infty$ and $(N_i)_{i=1}^\infty$ such that

(i) each $P_i$ is a compact polyhedron in $E^n$, $\dim P_i < k$,
(ii) each $N_i$ is a regular neighborhood of $P_i$ in $E^n$,
(iii) each $N_{i+1} \subset \text{int } N_i$, and
(iv) $X = \bigcap_{i=1}^\infty N_i$.

If $\dim X = k$ and $2k + 1 \leq n$, then the set of embeddings $f: X \to E^n$ such that $f(X)$ is in standard position is a dense $G_\delta$-subset of the set of maps of $X$ into $E^n$ [5, Theorem 3.3]. In this section we show that compacta in $E^n$ which satisfy ILC and have shape dimension less than $n - 2$ are in standard position.

For any pair $(A, B)$, the notation $\pi_i(A, B) = 0$ means that every map $f: (\Delta^i \times 0, \partial \Delta^i \times 0) \to (A, B)$ extends to a map $\tilde{f}: (\Delta^i \times [0, 1], \partial \Delta^i \times [0, 1] \cup \Delta^i \times 1) \to (A, B)$. ($\Delta^i$ denotes the standard $i$-simplex.)

**Lemma 1.** Let $X \subset E^n$ be a compactum satisfying ILC and let $k = \text{Sd}(X)$. Then $\pi_i(U, U - X) = 0$, $0 \leq i \leq n - k - 1$, for every compact neighborhood $U$ of $X$ in $E^n$.

**Proof.** It may be assumed that $U$ is connected because otherwise the following proof can be applied to each component of $U$. Since $\text{Sd}(X) = k$, there exists a compactum $Y$ with $\dim Y = k$ and $\text{Sh}(X) = \text{Sh}(Y)$. Embed $Y$ in $E^{2k+1}$ in standard position; say $Y = \bigcap_{i=1}^\infty N_i$ where each $N_i$ is a regular neighborhood of $P_i$ in $E^{2k+1}$, $\dim P_i < k$ and $N_{i+1} \subset \text{int } N_i$.

We first construct a convenient sequence of neighborhoods whose intersection is $X$. Let $(f_i, X, Y)_E^{E^{k+1}, E^n}$ and $(g_i, Y, X)_E^{E^{k+1}, E^n}$ be fundamental sequences which show that $X$ and $Y$ have the same shape [2, Theorem 2.4]. Choose an integer $j$ such that $g_i(N_i) \subset U$ for almost all $i$. Now choose a neighborhood $V$ of $X$ such that $f_i(V) \subset N_j$ and $g_i f_i|V \approx 1_V$ in $U$ for almost all $i$. It may be assumed that $g_i P_i$ is piecewise linear. Thus the inclusion map $\beta: V \subset U$ is homotopic in $U$ to a map of $V$ into $g_i(P_i)$. Inductively then we can construct a sequence of neighborhoods $V_i$ in $U$ and polyhedra $K_i$ such that $\dim K_i < k$, $V_i \cup K_i \subset V_{i-1}$, $\bigcap_{i=1}^\infty V_i = X$ and the inclusion map $V_i \to V_{i-1}$ is homotopic in $V_{i-1}$ to a map of $V_i$ into $K_i$.

Consider the universal cover $p: \widetilde{U} \to U$. Denote $p^{-1}(V_i)$ by $\widetilde{V}_i$ and $p^{-1}(X)$ by $\widetilde{X}$. We show that $H^q_c(\widetilde{X}) = 0$ for $q > k$. Let $f_i: V_i \times [0, 1] \to V_{i-1}$ be a homotopy such that $f_0 = 1_{V_i}$ and $f_1(V_i) \subset K_i$. The diagram

\[
\begin{array}{ccc}
\widetilde{U} & \xrightarrow{p} & U \\
\widetilde{V}_i \searrow & & \nearrow p \\
\end{array}
\]
commutes, so $f_\ast p|\overline{V}_i$ can be lifted to a homotopy $g_\ast$. Since $g_\ast(\overline{V}_i) \subset p^{-1}(K_i)$ and $p$ is a local homeomorphism, $\dim g_\ast(\overline{V}_i) \leq k$.

Let $\overline{V}_i^+$ and $\overline{X}^+$ denote the one-point compactifications of $\overline{V}_i$ and $\overline{X}$ respectively. Since $g_\ast$ is a proper map, $g_\ast$ can be extended to $\tilde{g}_\ast: \overline{V}_i^+ \to \overline{V}_i^+$. Hence the inclusion $\overline{V}_i^+ \hookrightarrow \overline{V}_i^{i+1}$ factors up to homotopy through a map into a space of dimension at most $k$. Thus the continuity axiom implies that $H^q(\overline{X}^+) = 0$ for $q > k$. Finally [13, Corollary 6.6.12] shows that $H^q(\overline{X}) = 0$, $q > k$.

Now Alexander duality [13, Theorem 6.9.10] gives $H_q(\tilde{U}, \tilde{U} - \overline{X}) \approx H^{n-q}(\overline{X}) = 0$ for $n - q > k + 1$. We look at the homology sequence of the pair $(\tilde{U}, \tilde{U} - \overline{X})$ and see that $H_0(\tilde{U} - \overline{X}) = 0$. So $\tilde{U} - \overline{X}$ is connected.

Let $V_j$ and $X_*$ denote the one-point compactifications of $V_j$ and $X$ respectively. Since $g_\ast$ is a proper map, $g_\ast$ can be extended to $\tilde{g}_\ast: \overline{V}_j^+ \to \overline{V}_j^+$.

Hence the inclusion $\overline{V}_j^+ \hookrightarrow \overline{V}_j^{j+1}$ factors up to homotopy through a map into a space of dimension at most $k$. Thus the continuity axiom implies that $H^q(\overline{X}^+) = 0$ for $q > k$. Finally [13, Corollary 6.6.12] shows that $H^q(\overline{X}) = 0$, $q > k$.

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Let $\pi_*(U, U - X) = 0$ since if $f: (\Delta^1 \times 0, \partial \Delta^1 \times 0) \to (U, U - X)$, $f$ can be lifted to $f': (\Delta^1 \times 0, \partial \Delta^1 \times 0) \to (\tilde{U}, \tilde{U} - \overline{X})$. $\tilde{U} - \overline{X}$ is connected, so $f'(0)$ and $f'(1)$ can be joined by an arc in $\tilde{U} - \overline{X}$. The resulting loop is null-homotopic in $\tilde{U}$. The projection down of this homotopy is the desired homotopy in $U$.

We next prove that $\pi_i(\tilde{U} - \overline{X}) = 0$. Let $f: S^1 \to \tilde{U} - \overline{X}$ be a loop. $f$ extends to $\tilde{f}: \Delta^2 \to \tilde{U}$. Consider $p\tilde{f}: (\Delta^2, S^1) \to (U, U - X)$. Choose $V \subset U$ to be a neighborhood of $X$ satisfying the inessential loops condition relative to $U$.

Now triangulate $\Delta^2$ so that each simplex whose image intersects $X$ is mapped into $V$. Using the fact that $\pi_0(V, V - X) = 0$ (by the same argument as was used to show that $\pi_0(U, U - X) = 0$), we can push the image of the $1$-skeleton of this triangulation off $X$. If $\sigma$ is a $2$-simplex in $\Delta^2$, $p\tilde{f}|\partial \sigma: \partial \sigma \to V - X$ and $p\tilde{f}|\partial \sigma \simeq 0$ in $V$, so $p\tilde{f}|\sigma$ may be replaced by a map of $\sigma$ into $U - X$ which agrees with $p\tilde{f}$ on $\partial \sigma$. Thus $p\tilde{f}|S^1 \simeq 0$ in $U - X$. Lifting this homotopy, we see that $f \simeq 0$ in $\tilde{U} - \overline{X}$.

Finally, we apply the relative Hurewicz Theorem [13, Theorem 7.5.4] and conclude that $\pi_i(\tilde{U}, \tilde{U} - \overline{X}) \approx H_i(\tilde{U}, \tilde{U} - \overline{X}) = 0$, $2 \leq i \leq n - k - 1$. The homotopy lifting property now can be used to show that $\pi_i(U, U - X) = 0$, $i \leq n - k - 1$.

**Lemma 2.** Let $X \subset E^n$ be a compactum such that $\text{Sd}(X) \leq n - 3$. Given a neighborhood $U$ of $X$ there exists a neighborhood $V$ of $X$ such that for any compact polyhedron $K \subset V$ with $\dim K \leq n - 3$ there is a polyhedron $P$ and a regular neighborhood $N$ of $P$ such that $K \subset \text{int} N \subset N \subset U$ and $\dim P \leq \text{Sd}(X)$.

**Proof.** The proof is by induction on $k = \dim K$. If $k \leq \text{Sd}(X)$, $P = K$ and $V = U$ will work; so it may be assumed that $k > \text{Sd}(X)$ and that the lemma is true for polyhedra of dimension less than $k$. Choose $V' \subset U$ using this inductive hypothesis. As before choose a neighborhood $V$ of $X$ in $V'$ and a polyhedron $P' \subset V'$ such that $\dim P' \leq \text{Sd}(X)$ and the inclusion $A \hookrightarrow V'$ is homotopic in $V'$ to a map of $V$ into $P'$.

Let $f: K \times [0, 1] \to V'$ be a homotopy such that $f_0 = 1_K$ and $f_1(K) \subset P'$.

By Zeeman’s Piping Lemma [15, Lemma 48], we may assume that there exists a polyhedron $J \subset K \times I$ such that

1. $S(J) \subset J$,
2. $\dim J \leq 2k - n + 2 \leq k - 1$,
3. $\dim \left( J \cap (K^{k-1} \times [0, 1]) \right) \leq 2k - n + 1 \leq k - 2$, and
4. $K \times [0, 1] \to J \cup (K^{k-1} \times [0, 1]) \cup K \times 1$. (Here $K^i$ denotes the $i$-skeleton of $K$.)
Let \( L \) be a \((k-2)\)-dimensional subpolyhedron of \( K \) such that \( L \supseteq K^{k-2} \) and \( L \times [0, 1] \supseteq J \cap (K^{k-1} \times [0, 1]) \). By induction it may be assumed that \( f(L \times [0, 1] \cup J) \cup P' \subseteq N \subseteq U \) where \( N \) is a regular neighborhood of some polyhedron \( P \) with dimension \( \leq \text{Sd}(X) \). It remains only to engulf \((k-1)\)- and \( k \)-simplexes of \( K \).

\[ K^{k-1} \times [0, 1] \searrow K^{k-2} \times [0, 1] \cup K^{k-1} \times 1 \cup (L \cap K^{k-1}) \times [0, 1]. \]

The image of the latter set is already contained in \( N \) and contains \( S(f|K^{k-1} \times [0, 1]) \). Following the image of this collapse, \( K^{k-1} \) can be engulfed with \( N \). Similarly \( K \times [0, 1] \searrow J \cup K^{k-1} \times [0, 1] \cup K \times 1 \), so \( N \) can be pushed out to cover all of \( K \).

**Lemma 3.** Let \( X \subset \mathbb{R}^n, n > 5 \), be a compactum with \( \text{Sd}(X) < n-3 \). Then \( X \) satisfies ILC if and only if \( X \) is in standard position.

**Proof.** It suffices to show that given a neighborhood \( U \) of \( X \) there exists a polyhedron \( P \) in \( U \) with \( \text{dim} P \leq \text{Sd}(X) \) and a regular neighborhood \( N \) of \( P \) such that \( X \subset \text{int} N \subset N \subset U \). Let \( V \subset U \) be given by Lemma 2 and let \( M \) be a compact PL manifold neighborhood of \( X \) in \( V \). Denote the \((n-3)\)-skeleton of \( M \) by \( M^{n-3} \) and the dual 2-skeleton by \( M_2^* \). By Lemma 2 there exists a polyhedron \( P \) with \( \text{dim} P \leq \text{Sd}(X) \) and a regular neighborhood \( N \) of \( P \) such that \( M^{n-3} \subset \text{int} N \subset N \subset U \).

By Lemma 1 and Stallings' engulfing theorem [14], there exists a PL homeomorphism \( h_1: \text{int} M \to \text{int} M \) with compact support such that \( h_1(\text{int} M - X) \supseteq M_2^* \cap \text{int} M \). Extend \( h_1 \) via the identity to \( U \). Let \( h_2 \) be a homeomorphism of \( U \) which pushes \( N \) across the join structure between \( M^{n-3} \) and \( M_2^* \) until \( M \subset h_1(U - X) \cup h_2(\text{int} N) \). Then \( h_1^{-1} h_2(N) \) is the regular neighborhood we are looking for.

### 4. Proofs of Theorems 1 and 2

In this section we complete the proof of Theorem 1 and prove Theorem 2. The following lemma is used to keep an inductive argument going in the proof of Theorem 1.

**Lemma 4.** Let \( X, Y \subset \mathbb{R}^n, n > 5 \), be compacta in standard position with shape dimensions in the trivial range with respect to \( n \) and let \( \{f_i, X, Y\} \) and \( \{f'_i, Y, X\} \) be fundamental sequences which are homotopy inverse to one another. Let \( U_0 \) be a neighborhood of \( X \) and \( h \) be a PL homeomorphism of \( \mathbb{R}^n \) such that \( Y \subset h(U_0) \) and such that there exists a neighborhood \( W_0 \) of \( Y \) with \( h^{-1}|W_0 = f_i|W_0 \) in \( U_0 \) for almost all \( i \). Then for every open set \( V_0 \subset \mathbb{R}^n \) there exists a PL homeomorphism \( q \) of \( \mathbb{R}^n \) such that \( q|E^n - U_0 = h|E^n - U_0 \) \( X \subset q^{-1}(V_0) \), and \( q|U_1 = f_i|U_1 \) in \( V_0 \) for almost all \( i \) where \( U_1 \) is some neighborhood of \( X \).

**Proof.** Assume that \( X = \bigcap_{j=1}^\infty M_j \) where each \( M_j \) is a regular neighborhood of a compact polyhedron \( L_j \) and that \( \text{dim} L_j \leq \text{Sd}(X) \). Choose neighborhoods \( V \subset V_0 \) of \( Y \) and \( U \subset U_0 \) of \( X \) and an integer \( i_0 \) such that \( h^{-1}|V = f'_i|V \) in \( U_0 \), \( f_i|U = f_{i+1}|U \) in \( V \), and \( f'_i|f_i|U = 1_U \) in \( U_0 \) for \( i > i_0 \). Let \( j \) be an integer large enough so that \( M_j \subset U \). \( f_{i_0}|L_j \) can be approximated by a PL embedding \( \hat{f}_j \).

Notice that \( h^{-1} \circ \hat{f} \approx f'_{i_0} \circ \hat{f} \approx f'_{i_0} \circ f_{i_0}|L_j \approx 1_{L_j} \) in \( U_0 \).

So \( \hat{f} = h|L_j \) in \( h(U_0) \). Hence there exists a PL homeomorphism \( r \) of \( E^n \) which is the identity outside \( h(U_0) \) and such that \( rh|L_j = \hat{f} \) [6]. It may be assumed
that \( rh(M_j) \subset V \). Taking \( q = rh \) and \( U_1 = M_j \) gives the desired conclusion.

**Proof of Theorem 1.** Suppose that \( Sh(X) = Sh(Y) \). A homeomorphism of \( E^n - X \) onto \( E^n - Y \) can be constructed using the technique of [5, Lemma 4.2]. Lemma 4 above replaces Lemma 4.1 of [5]. Now suppose that \( E^n - X \cong E^n - Y \). There exist \( X' \) and \( Y' \) with dimension in the trivial range satisfying \( Sh(X) = Sh(X') \) and \( Sh(Y) = Sh(Y') \). It may be assumed that \( X' \) and \( Y' \) are embedded in \( E^n \) as 1-ULC subsets [7, Lemmas 2 or 5, §3]. The first part of the theorem implies that \( E^n - X' \cong E^n - Y' \), so \( Sh(X) = Sh(Y) \) by [5] again.

**Proof of Theorem 2.** The implication \((iii) \Rightarrow (ii)\) is obvious and \((ii) \Rightarrow (iii)\) is exactly Corollary 1.3 of [9]. Theorem 1 gives \((iii) \Rightarrow (i)\). Our proof that \((i) \Rightarrow (iii)\) actually establishes a stronger result which we state as Theorem 3.

**Theorem 3.** Let \( X, Y \subset E^n \) be compacta and let \( A, B \) be compact connected abelian topological groups with \( Sh(X) = Sh(A) \) and \( Sh(Y) = Sh(B) \). Then \( E^n - X \cong E^n - Y \) implies \( A \approx B \).

**Proof.** Suppose \( E^n - X \cong E^n - Y \). Then by Alexander duality [13, Theorem 6.2.16] \( H^1(X) \cong H^1(Y) \); hence \( H^1(A) \cong H^1(B) \) [10, Theorem 16]. Therefore \( char A \approx char B \) [8, Theorem 1.4] and so Pontryagin duality [11, Theorem 52] shows that \( A \approx B \).

### References