THE EXISTENCE OF DUAL MODULES

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Abstract. In this note we show that a Noetherian module has a dual module if and only if it satisfies AB5*. A connection between completeness and AB5* is also established.

In this note we relate completeness, quasi-completeness, the AB5* condition, and duality. The main result is that a Noetherian R-module has a dual module if and only if it satisfies AB5*.

Throughout this note R will denote a commutative ring with identity and all modules will be unitary. The terms local and semilocal will carry the Noetherian hypothesis. We use L(A) or LR(A) to denote the lattice of R-submodules of A.

An R-module B is said to be a dual of an R-module A if there exists an order reversing lattice isomorphism θ: L(A) → L(B) satisfying θ(JN) = θ(N): J for all R-submodules N of A and all ideals J of R.

Any R-module A satisfies the so-called AB5 condition: for any submodule B and any ascending chain \( \{ B_\alpha \} \) of submodules of A, \( B \cap (\bigcup \alpha B_\alpha) = \bigcup \alpha (B \cap B_\alpha) \). A satisfies the dual condition AB5* if for any submodule B and any descending chain \( \{ B_\alpha \} \) of submodules, it follows that \( B + (\bigcap \alpha B_\alpha) = \bigcap \alpha (B + B_\alpha) \). Not every module satisfies AB5*; for example, \( \mathbb{Z} \), the integers, does not. However, any module having a dual necessarily satisfies AB5*. We show that for Noetherian modules, the converse is also true. We first show that the condition AB5* is closely related to completeness.

Let R be a semilocal ring with Jacobson radical J and let A be a finitely generated R-module. If A is complete in the J-adic topology, it is well known [6] that A satisfies the condition

\[ (*) \text{ For any descending chain } \{ B_n \}_{n=1}^\infty \text{ of submodules of } A \text{ and any integer } k, \text{ there exists an integer } n(k) \text{ such that } B_{n(k)} \subseteq (\bigcap_{n=1}^\infty B_n) + J^kA. \]

A finitely generated module over a semilocal ring will be called quasi-complete if it satisfies (*).

The first theorem relates the concepts of quasi-completeness and AB5*.

Theorem 1. Let R be a semilocal ring and A a finitely generated R-module. Then A satisfies AB5* if and only if it is quasi-complete.

Proof. Suppose A satisfies AB5*. Let \( \{ B_n \}_{n=1}^\infty \) be a countable descending chain of submodules of A and let k be a fixed integer. Then

\[ J^kA + \bigcap_{n=1}^\infty B_n = \bigcap_{n=1}^\infty (J^kA + B_n) = J^kA + B_{n(k)} \]
for some integer $n(k)$ since $A$ satisfies $AB5^*$ and $A/J^kA$ is Artinian. Thus $B_n(k) \subseteq \bigcap_{n=1}^{\infty} B_n + J^kA$, so $A$ is quasi-complete. Conversely, suppose that $A$ is quasi-complete. Let $\{B_n\}_{n=1}^{\infty}$ be a countable descending chain in $A$, $B = \bigcap_{n=1}^{\infty} B_n$, and let $C$ be a submodule of $A$. We show that $C + B = \bigcap_{n=1}^{\infty} (C + B_n)$. For fixed $k$, by quasi-completeness, there exists an integer $n(k)$ such that $B_n(k) \subseteq B + J^kA$, and hence $C + B_n(k) \subseteq C + B + J^kA$. We may assume $n(k) \to \infty$ as $k \to \infty$. Hence

$$\bigcap_{n=1}^{\infty} (C + B_n) = \bigcap_{n=1}^{\infty} (C + B_n(k)) \subseteq \bigcap_{k=1}^{\infty} (B + C + J^kA) = B + C$$

by the Krull Intersection Theorem. The reverse containment is always true. The result now follows since any chain in $A$ is countable [2].

We next relate completeness and quasi-completeness. Let $R$ be semilocal with Jacobson radical $J$ and let $A$ be a finitely generated $R$-module. $L(A)$, the lattice of submodules of $A$ has a natural metric $d$ defined on it by $d(C, D) = 2^{-n}$ if $C + J^nA = D + J^nA$ but $C + J^{n+1}A \neq D + J^{n+1}A$. The next theorem is due to E. W. Johnson [3].

**Theorem 2.** Let $R$ be a semilocal ring and $A$ a finitely generated $R$-module. Then the following are equivalent:

1. the metric $d$ on $L(A)$ is complete,
2. $A$ is quasi-complete,
3. the map $L_R(A) \to L_R(\hat{A})$ given by $N \to \hat{R}N$ (where $\hat{\cdot}$ denotes the $J$-adic completion) is surjective (and hence a lattice isomorphism).

We remark that while any complete module is quasi-complete, a quasi-complete module need not be complete. For example, any $D \neq R$ is quasi-complete. More generally a one-dimensional local domain is quasi-complete if and only if it is analytically irreducible. The ring $k[X, Y]_{(X, Y)}$, $k$ a field, is not quasi-complete.

The main theorem requires the following

**Lemma.** Let $R$ be a Noetherian ring and $A$ a finitely generated $R$-module satisfying $AB5^*$. Then $\text{Supp}(A)$ contains only finitely many maximal elements; actually each $P \in \text{Ass}(A)$ is contained in a unique maximal element of $\text{Supp}(A)$.

**Proof.** Since $\text{Ass}(A)$ is finite, the second statement implies the first. For $P \in \text{Ass}(A)$, $R/P$ is isomorphic to a submodule of $A$ and hence satisfies $AB5^*$ as an $R$-module and hence as a ring. Thus we are reduced to showing that a Noetherian domain $R$ satisfying $AB5^*$ must be local. Suppose not, say $P$ and $Q$ are distinct maximal ideals in $R$. Now $\{P^n\}_{n=1}^{\infty}$ is a descending chain of ideals in $R$ and $\bigcap_{n=1}^{\infty} P^n = 0$ by the Krull Intersection Theorem. Hence $Q + \bigcap_{n=1}^{\infty} P^n = R$. However, for every $n$, $Q + P^n = R$, so $\bigcap_{n=1}^{\infty} (Q + P^n) = R$. Thus $R$ must be local.

Finally, we require the theory of duality between Noetherian and Artinian modules over a complete local (or semilocal) ring given by Matlis [4] and [5]. (Also see [7] for an introduction into injective modules and duality.) Briefly, let $R$ be a complete semilocal ring with Jacobson radical $J$. There is a perfect duality between Noetherian and Artinian $R$-modules given by the functor $\text{Hom}_R(-, E(R/J))$ where $E(R/J)$ is the injective envelope of $R/J$. Also for $R$...
semilocal, but not necessarily complete, and for \( A \) an Artinian \( \hat{R} \)-module, the \( R \)-submodules and \( \hat{R} \)-submodules coincide and hence \( A \) is also Artinian as an \( R \)-module.

Theorem 3. For a finitely generated module \( A \) over a Noetherian ring \( R \), the following are equivalent:

1. \( A \) has a dual,
2. \( A \) satisfies \( AB5^{*} \),
3. \( A \) is quasi-complete as an \( \bar{R} = R/\text{ann}(A) \)-module.

Proof. It is clear that (1) implies (2). Suppose \( A \) satisfies \( AB5^{*} \). Then \( A \) satisfies \( AB5^{*} \) as an \( \bar{R} \)-module. By the previous lemma, \( \bar{R} \) is semilocal. By Theorem 1, \( A \) is quasi-complete as an \( \bar{R} \)-module. It remains to show that (3) implies (1). By change of rings, it suffices to show that \( A \) has an \( \bar{R} \)-module dual. Thus we may replace \( R \) by \( \bar{R} \) and assume that \( R \) is semilocal. By Theorem 2, the map \( L_{R}(A) \to L_{\bar{R}}(\hat{A}) \) given by \( N \to \bar{R}N \) is a lattice isomorphism which preserves scalar product (i.e., \( \bar{R}(JN) = J(\bar{R}N) \)). Now as an \( \bar{R} \)-module, \( \hat{A} \) has a dual, namely, \( B = \text{Hom}_{\bar{R}}(\hat{A}, E(\bar{R}/J)) \). Since \( B \) is Artinian as an \( \bar{R} \)-module, it follows that the \( R \)-submodules of \( B \) coincide with the \( \bar{R} \)-submodules of \( B \). Hence \( B \) is actually an \( R \)-module dual of \( A \).

We have shown that a Noetherian module has a dual if and only if it satisfies \( AB5^{*} \). The hypothesis that the module be Noetherian cannot be deleted. Any Artinian module satisfies \( AB5^{*} \); however, it is easily seen that the abelian group \( \mathbb{Z}_{p^{\infty}} \) (\( p \) a prime) does not have a \( Z \)-module dual. In fact, a result of Baer [1] states that an abelian group has a dual if and only if it is torsion and every primary component is finitely generated.

References


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