A CHARACTERIZATION OF $C^\infty$-SUFFICIENT $k$-JETS

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Abstract. We improve some results of Mather and Arnold and find several necessary and sufficient conditions of sufficiency of $k$-jets. As a corollary we prove that the set of $C^\infty$-sufficient $k$-jets is a semialgebraic subset of the space of $k$-jets of $C^\infty$ mappings $F: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$.

1. Introduction. Let $\mathcal{E}$ denote the local $\mathbb{R}$-algebra of germs at $0 \in \mathbb{R}^n$ of all $C^\infty$ functions from $\mathbb{R}^n$ to $\mathbb{R}$, and let $\mathfrak{M}$ be the unique maximal ideal of $\mathcal{E}$. For a given $f \in \mathcal{E}$, let $I(f)$ denote the ideal in $\mathcal{E}$ generated by $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$. The space of $k$-jets of $C^\infty$ mappings $h: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is denoted by $J^k(n, p)$ and, by definition, $J^k(n, 1) = J^k$. $j^k(f)$ is the $k$-jet of $f$. Let $\mathcal{R}_l$ denote the group of germs of $C^\infty$ diffeomorphisms $h: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, such that $j^l(h)$ coincides with the $l$-jet of the identity mapping $\mathbb{R}^n$ onto itself. Clearly $\mathcal{R}_0$ is the set of all local diffeomorphisms around 0 in $\mathbb{R}^n$.

Definition. A $k$-jet $w \in J^k$ is called $\mathcal{R}_l$ sufficient ($0 \leq l \leq k$) if for any $f \in \mathcal{E}$ such that $j^k(f) = w$, there exists $h \in \mathcal{R}_l$ such that $f \circ h = w$. A $\mathcal{R}_0$ sufficient jet is called sufficient.

The following theorems are known:

Theorem I (Mather [4], Arnold [1]). Let $f \in \mathfrak{M}$ and $1 \leq l \leq k$. Suppose $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(f)$. Then the $k$-jet $j^k(f)$, of $f$ is $\mathcal{R}_l$ sufficient.

Theorem II (Mather [3]). Let $f \in \mathfrak{M}$ and $0 \leq l \leq k$. If the $k$-jet $j^k(f)$ is $\mathcal{R}_l$ sufficient, then $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(f)$.

In particular for $1 \leq l \leq k$, a $k$-jet $w \in J^k$ is $\mathcal{R}_l$ sufficient if and only if $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(w)$. As a corollary one can observe that for $w \in J^k$, the condition $\mathfrak{M}^{k+1} \subset \mathfrak{M}^2 I(w)$ implies the sufficiency of $w$; and if $w$ is sufficient, then $\mathfrak{M}^{k+1} \subset \mathfrak{M} I(w)$. However, none of these conditions characterizes the sufficiency of $k$-jets.

Counterexample 1 (communicated by J. Robbin). For $w = x^2 + 2xy^2 \in J^3$ we have $\mathfrak{M}^4 \subset \mathfrak{M} I(w)$, but the jet $w$ is not sufficient. In fact, $0 \in \mathfrak{M}^2$ is an isolated critical point for $w$, but not for $w + y^4$.

Counterexample 2. Let $w(x, y) = x^3 + xy^3 \in J^4$. Arnold proved [1, p. 12] that the jet $w$ is sufficient (it also follows from our Theorem 2). However, one can verify that $y^5 \notin \mathfrak{M}^2 I(w)$ and, hence, $\mathfrak{M}^5 \notin \mathfrak{M}^2 I(w)$.

Here we shall improve the results above, and find necessary and sufficient...
conditions of sufficiency of $k$-jets (Theorem 2). We show also that the set of sufficient $k$-jets is a semialgebraic subset of $J^k$ (Theorem 3).

2. Results. Put $\mathbb{R}_l^k = \{ j^k(h) \in J^k(n,n); h \in \mathbb{R}_l \}$, $l = 0, \ldots, k$. It is clear that one can consider $J^k$ as a finite dimensional vector space over $\mathbb{R}$ and $\mathbb{R}_0^k$ (respectively, $\mathbb{R}_l^k$, $l = 1, \ldots, k$) as an open subset (respectively, an affine subspace) of $J^k$. It is well known that $\mathbb{R}_l^k$ has the structure of a Lie group, and as such it acts smoothly on the left on $J^k$, by the formula $j^k(h) \cdot j^k(f) = j^k(f \circ h^{-1})$. Hence, any orbit of $\mathbb{R}_l^k$ in $J^k$ is a submanifold in $J^k$. For $f \in \mathcal{W}$, denote by $V^k(f)$ the orbit of $\mathbb{R}_l^k$ in $J^k$ passing through $j^k(f)$; codim $V^k(f)$ is the codimension of $V^k(f)$ in $J^k$.

Define $\pi: J^{k+1} \ni z \rightarrow j^k(z) \in J^k$. For $f \in \mathcal{W}$ and $0 \leq s \leq q$, let

$$ A^q_s(f): (\mathcal{W}^{s+1}/\mathcal{W}^{q+1})^n \rightarrow \mathcal{W}^{s+1}/\mathcal{W}^{q+1} $$

be the $\mathbb{R}$-linear mapping defined by the formula

$$ A^q_s(f)(\bar{u}_1, \ldots, \bar{u}_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \bar{u}_i, $$

where $\bar{u}$ denotes the equivalence class of $u \in \mathcal{W}$ modulo $\mathcal{W}^{q+1}$. Note that $j^{q-s}(f) = j^{q-s}(g)$ implies $A^q_s(f) = A^q_s(g)$.

**Theorem 1.** Let $w \in J^k$ and $1 \leq l \leq k$. The following conditions are equivalent:

1. The $k$-jet $w$ is $\mathbb{R}_l$ sufficient;
2. $\mathcal{W}^{k+1} \subset \mathcal{W}^{l+1}(w)$;
3. $\mathcal{W}^{k+1} \subset \mathcal{W}^{l+1}(f)$ for each $f \in \mathcal{W}$ with $j^k(f) = w$;
4. $\mathcal{W}^{k+1} \subset \mathcal{W}^{l+1}(f)$ for some $f \in \mathcal{W}$ with $j^k(f) = w$;
5. $\forall s > k$, codim $V^k(f)$ = codim $V^s(f)$;
6. $\exists s > k \ni$ codim $V^k(f)$ = codim $V^s(f)$.

**Corollary 1.** For $f \in \mathcal{W}$ and $1 \leq l \leq k$ the following conditions are equivalent:

1. $j^k(f)$ is $\mathbb{R}_l$ sufficient;
2. $\forall s > k$, codim $V^k(f)$ = codim $V^s(f)$;
3. $\exists s > k \ni$ codim $V^k(f)$ = codim $V^s(f)$.

**Theorem 2.** For a given $k$-jet $w \in J^k$ the following conditions are equivalent:

a. The $k$-jet $w$ is sufficient;

b. $\mathcal{W}^{k+1} \subset \mathcal{W}^{l+1}(z)$ for each $z \in \pi^{-1}(w)$;

c. $\mathcal{W}^{k+1} \subset \mathcal{W}^{l+1}(f)$ for each $f \in \mathcal{W}$ with $j^k(f) = w$;

d. codim $V^k(z)$ = codim $V^0(z)$ for each $z \in \pi^{-1}(w)$;

e. codim $V^k(f)$ = codim $V^0(z)$ for each $f \in \mathcal{W}$ with $j^k(f) = w$.

From this we have
Corollary 2. If a k-jet $j^k(f)$ of $f \in \mathcal{M}$ is sufficient, then \( \text{codim } V_{0}^s(f) = \text{codim } V_{0}^s(f) \) for each $s \geq k$.

Observe that, given $1 \leq l \leq k$ and $f, g \in \mathcal{M}$ with $j^k(f) = j^k(g)$, we have $\mathcal{M}^{k+1} \subset \mathcal{M}^{l+1}(f)$ if and only if $\mathcal{M}^{k+1} \subset \mathcal{M}^{l+1}(g)$ (by Nakayama’s lemma). If $l = 0$, the last equivalence does not hold (see Theorem 2 and Counterexample 1).

Theorem 3. The set $\Omega^k_l$ of $\mathcal{R}_l$ sufficient $k$-jets is a semialgebraic subset of $\mathcal{J}^k$, $0 \leq l \leq k$.

One can estimate the codimension of $\Omega^k_l$ in $\mathcal{J}^k$. Given $k, l, p \in \mathbb{N}$, $k \geq p$, put $\omega(k, l) = \sup(r \in \mathbb{N}: k \geq (r + l)^n + l + 1)$, and denote by $\pi_{k, p}: \mathcal{J}^k \to \mathcal{J}^p$ the natural linear projection of $\mathcal{J}^k$ onto $\mathcal{J}^p$.

Proposition 1. $\text{codim } \Omega^k_l \leq \text{dim } \ker \pi_{k, \omega(k, l)}$ for $l = 1, \ldots, k$ and $\text{codim } \Omega^k_0 \leq \text{codim } \Omega^k_1$.

3. Proofs. Put $\pi^k: \mathcal{M} \ni g \to j^k(g) \in \mathcal{J}^k$. We need the following:

Lemma 1 [3]. Let $f \in \mathcal{M}$ and $0 \leq l \leq k$. The linear subspace $\pi^k(\mathcal{M}^{l+1}(f))$ is the tangent space of $\mathcal{V}^k(f)$ at $j^k(f)$.

Lemma 2. For given $f \in \mathcal{M}$ and $0 \leq l \leq k$, the following conditions are equivalent:

(i) $\mathcal{M}^{l+1} \subset \mathcal{M}^{l+1}(f)$;
(ii) $\text{rank } A^{l+1}_f(f) = (l + k) + \text{rank } A^k_f(f)$;
(iii) $\text{codim } V_{i}^{l+1}(f) = \text{codim } V_{i}^{l+1}(f)$.

Proof. Observe that, by Nakayama’s lemma, (i) is equivalent to

\[ (*) \quad \text{dim } R(\mathcal{M}^{l+1}(f)/\mathcal{M}^{l+1}) = \text{dim } R(\mathcal{M}^{k+1}(f)/\mathcal{M}^{k+1}). \]

Condition (*) is trivially equivalent to

\[ (**) \quad \text{dim } R((\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1}) = \text{dim } R((\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1}). \]

The well-known fact that $\mathcal{R}$-linear spaces $(\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1}$ and $(\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+2}$ are isomorphs, and (**) imply (i) $\iff$ (ii), because $(\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1}$ is the image of $A^k(f)$ and $\text{dim } R((\mathcal{M}^{k+1} + \mathcal{M}^{k+2} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1})$.

Lemma 1 implies that $\mathcal{R}$-linear spaces $(\mathcal{M}/\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))$ and $\mathcal{J}^k/\mathcal{V}_{i}^{k}(f)$ are isomorphs, $z = j^k(f)$. It is clear that (i) is equivalent to

\[ (***) \quad \text{dim } R((\mathcal{M}/\mathcal{M}^{k+1} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1}) = \text{dim } R((\mathcal{M}/\mathcal{M}^{k+2} + \mathcal{M}^{l+1}(f))/\mathcal{M}^{k+1}). \]

Combining these facts one obtains (i) $\iff$ (iii).

Proof of Theorem 1. (1) $\iff$ (2) $\iff$ (2') $\iff$ (2'') follows from Theorems I and II. (3) $\iff$ (3') $\iff$ (3'') is an immediate consequence of the definition of $A^k(f)$.

(2) $\iff$ (3) $\iff$ (4) follows from Lemma 2.

(4) $\iff$ (4') $\iff$ (4''). The fact that $j^k(w) = j^k(f)$ implies $\mathcal{M}^{k+s+1} + \mathcal{M}^{l+1}(w) = \mathcal{M}^{k+s+1} + \mathcal{M}^{l+1}(f)$, $s = 0, 1$, because $l \geq 1$. Equalities $\text{codim } V_{i}^{k+s}(f) = \text{codim } V_{i}^{k+s}(f)$, $s = 0, 1, l \geq 1$, follow from the last observation, since $\mathcal{R}$-
linear spaces \((\mathcal{M}/\mathcal{M}^{k+1} + \mathcal{M}^{l+1}I(f))\) and \(J^{k+s}/T^k I V^{k+s}(f)\) are isomorphs, \(f \in \mathcal{M}, z = j^{k+s}(f), s = 0, 1, l \geq 1.\)

**Proof of Corollary 1.** (i) \(\Rightarrow\) (ii) and (iii) \(\Rightarrow\) (i) follow by induction from Theorem 1 (observe that if \(j^k(f)\) is \(\mathcal{G}_j\) sufficient, then \(j^q(f)\) is also \(\mathcal{G}_j\) sufficient for \(q \geq k\).) Implication (ii) \(\Rightarrow\) (iii) is trivial.

**Proof of Theorem 2.** (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (b') follow, respectively, from Theorem II and Nakayama’s lemma.

(b) \(\Rightarrow\) (a). First recall the following theorem of Mather [3, IV]: Let \(C^\infty\) Lie group \(G\) act on \(C^\infty\) manifolds \(M\) and \(N\). Suppose that \(C^\infty\) mapping \(F: M \rightarrow N\) is \(G\)-submersion (i.e. \(F\) is a submersion and \(F(xg) = F(x)g\) for \(x \in M, g \in G\)). Let \(y \in N\) and \(V = F^{-1}(y)\). If \(V\) is a connected submanifold of \(M\), then \(V\) is contained in a single orbit of the group \(G\) in \(M\) if and only if \(T_x V \subset T_x(xG)\) for each \(x \in V\).

We shall use this theorem in the following situation. The group \(\mathcal{G}_0^{k+1}\) acts on \(J^{k+1}\). Define the action of \(\mathcal{G}_0^{k+1}\) on \(J^k\) by \(J^k \times \mathcal{G}_0^{k+1} \ni (z, h) \rightarrow j^k(z \circ h^{-1}) \in J^k\). The mapping \(\pi: J^{k+1} \ni z \rightarrow j^k(z) \in J^k\) is a \(\mathcal{G}_0^{k+1}\)-submersion, and the set \(U^{k+1} = \pi^{-1}(w)\), as an affine subspace of \(J^{k+1}\), is a connected submanifold of \(J^{k+1}\). Consider \(w\) as an element of \(J^{k+1}\). The tangent space of \(U^{k+1}\) at \(w\) is the space \(\pi^{k+1}(\mathcal{M}^{k+1})\), and by Lemma 1 the tangent space of \(V^{k+1}\) at \(w\) equals \(\pi^{k+1}(\mathcal{M} I(w))\). By condition (b) we have \(\mathcal{M}^{k+1} \subset \mathcal{M} I(w)\) and, hence, also, \(\pi^{k+1}(\mathcal{M}^{k+1}) \subset \pi^{k+1}(\mathcal{M} I(w))\). Observe that \(w \in V^{k+1}(w) \cap U^{k+1}\) and, hence, by Mather’s theorem \(U^{k+1}\) is contained in \(V^{k+1}(w)\). Now we can prove that \(w \in J^k\) is a sufficient jet. Take \(f \in \mathcal{M}\) with \(j^k(f) = w\). There exists a germ \(h \in \mathcal{G}_0\) such that \(j^{k+1}(f) = j^{k+1}(w \circ h)\), because \(j^{k+1}(f) \in U^{k+1} \subset V^{k+1}(w)\). (b) implies \(\mathcal{M}^{k+2} \subset \mathcal{M} I(w \circ h)\), which is equivalent to the \(\mathcal{G}_j\) sufficiency of \(j^{k+1}(w \circ h) \in J^{k+1}\). Thus there exists a germ \(h' \in \mathcal{G}_0\) such that \(f = w \circ h \circ h'\), which proves the sufficiency of \(k\)-jet \(w\).

(b) \(\Leftrightarrow\) (c) and (b) \(\Leftrightarrow\) (d) follow from Lemma 2. (c) \(\Leftrightarrow\) (c') is, as earlier, an easy consequence of the definition of \(A^0_k(z)\). A proof of (d) \(\Leftrightarrow\) (d') is similar to one for Theorem 1, (4) \(\Leftrightarrow\) (4').

Corollary 2 can be deduced from Theorem 2 by simple induction on \(s\).

**Proof of Theorem 3.** Let \(1 \leq l \leq k\). By Theorem 1 we have

\[ \Omega^k_l = \left\{ w \in J^k : \left( \begin{array}{c} n + k \\ k + 1 \end{array} \right) + \text{rank } A^k_l(w) = \text{rank } A^{k+1}_l(w) \right\}. \]

Hence \(\Omega^k_l\) is a semialgebraic subset of \(J^k\), \(1 \leq l \leq k\). Now put

\[ \Delta^k_{l+1} = \{ z \in J^{k+1} : \mathcal{M}^{k+1} \subset \mathcal{M} I(z) \}. \]

The set \(\Delta^k_{l+1}\) is a semialgebraic subset of \(J^{k+1}\) because, by Lemma 2,

\[ \Delta^k_{l+1} = \left\{ z \in J^{k+1} : \left( \begin{array}{c} n + k \\ k + 1 \end{array} \right) + \text{rank } A^k_l(z) > \text{rank } A^{k+1}_l(z) \right\}. \]

Theorem 2 implies that \(\pi(\Delta^k_{l+1}) = J^{k+1}/\Omega^k_0\). Hence by the Seidenberg-Tarski theorem [5], the set \(J^{k+1}/\Omega^k_0\) is a linear projection of a semialgebraic set—is semialgebraic. Clearly \(\Omega^k_0\) is also a semialgebraic set and Theorem 3 is proved.

**Proof of Proposition 1.** Inequality \(\text{codim } \Omega^k_s \leq \text{codim } \Omega^k_k\) is obvious. Let
$1 \leq l \leq k$. Fix $k \in \mathbb{N}$ and put $q = (k + 1)^n + l + 1$. Observe that for $w \in J^k$ we have $\pi^{-1}_{q,k}(w) \cap \Omega^q_l \neq \emptyset$. If $0 \in \mathbb{R}^n$ is a regular point of $w$, this is trivial. In the other case, if $0$ is a critical point of $w$, we can choose a homogeneous polynomial $h$ of degree $k + 1$ such that $0 \in \mathbb{C}^n$ is an isolated critical point of $w + h$, where $w + h$ is the complexification of $w + h$. A theorem of Samuel [2] implies $\mathcal{M}^{(k+1)^n} \subset I(w + h)$, and hence $j^q(w + h) \in \pi^{-1}_{q,h}(w) \cap \Omega^q_l$ (Theorem 1). This means that $\pi_{k,\omega(k,l)}(\Omega^k_l) = j_{\omega(k,l)}$ and, hence, $\text{codim } \Omega^k_l \leq \dim \ker \pi_{k,\omega(k,l)}$.

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