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A CHARACTERIZATION OF C^∞-SUFFICIENT k-JETS

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ABSTRACT. We improve some results of Mather and Arnold and find several necessary and sufficient conditions of sufficiency of k-jets. As a corollary we prove that the set of C^∞-sufficient k-jets is a semialgebraic subset of the space of k-jets of C^∞ mappings F: $(R^n, 0) \to (R, 0)$.

1. **Introduction.** Let $\mathcal{E}$ denote the local $\mathbf{R}$-algebra of germs at $0 \in \mathbf{R}^n$ of all C^∞ functions from $\mathbf{R}^n$ to $\mathbf{R}$, and let $\mathfrak{M}$ be the unique maximal ideal of $\mathcal{E}$. For a given $f \in \mathcal{E}$, let $I(f)$ denote the ideal in $\mathcal{E}$ generated by $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$. The space of k-jets of C^∞ mappings h: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is denoted by $J^k(n,p)$ and, by definition, $J^k(n,1) = J^k$; $j^k(f)$ is the k-jet of f. Let $\mathfrak{R}_l$ denote the group of germs of C^∞ diffeomorphisms h: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, such that $j^l(h)$ coincides with the l-jet of the identity mapping $\mathbf{R}^n$ onto itself. Clearly $\mathfrak{R}_0$ is the set of all local diffeomorphisms around 0 in $\mathbf{R}^n$.

DEFINITION. A k-jet $w \in J^k$ is called $\mathfrak{R}_l$ sufficient $(0 \leqslant l \leqslant k)$ if for any $f \in \mathcal{E}$ such that $j^k(f) = w$, there exists $h \in \mathfrak{R}_l$ such that $f \circ h = w$. A $\mathfrak{R}_0$ sufficient jet is called sufficient.

The following theorems are known:

THEOREM I (MATHER [4], ARNOLD [1]). *Let $f \in \mathfrak{M}$ and $1 \leqslant l \leqslant k$. Suppose $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(f)$. Then the k-jet $j^k(f)$, of f is $\mathfrak{R}_l$ sufficient.*

THEOREM II (MATHER [3]). *Let $f \in \mathfrak{M}$ and $0 \leqslant l \leqslant k$. If the k-jet $j^k(f)$ is $\mathfrak{R}_l$ sufficient, then $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(f)$.*

In particular for $1 \leqslant l \leqslant k$, a k-jet $w \in J^k$ is $\mathfrak{R}_l$ sufficient if and only if $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(w)$. As a corollary one can observe that for $w \in J^k$, the condition $\mathfrak{M}^{k+1} \subset \mathfrak{M}^2 I(w)$ implies the sufficiency of w; and if w is sufficient, then $\mathfrak{M}^{k+1} \subset \mathfrak{M} I(w)$. However, none of these conditions characterizes the sufficiency of k-jets.

Counterexample 1 (*communicated by J. Robbin*). For $w = x^2 + 2xy^2 \in J^3$ we have $\mathfrak{M}^4 \subset \mathfrak{M} I(w)$, but the jet w is not sufficient. In fact, $0 \in \mathbf{R}^2$ is an isolated critical point for w, but not for $w + y^4$.

Counterexample 2. Let $w(x,y) = x^3 + xy^3 \in J^4$. Arnold proved [1, p. 12] that the jet w is sufficient (it also follows from our Theorem 2). However, one can verify that $y^5 \notin \mathfrak{M}^2 I(w)$ and, hence, $\mathfrak{M}^5 \not\subset \mathfrak{M}^2 I(w)$.

Here we shall improve the results above, and find necessary and sufficient

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conditions of sufficiency of k-jets (Theorem 2). We show also that the set of sufficient k-jets is a semialgebraic subset of J^k (Theorem 3).

2. **Results.** Put $\mathfrak{R}_l^k = \{j^k(h) \in J^k(n,n)\colon h \in \mathfrak{R}_l\}$, $l = 0, \ldots, k$. It is clear that one can consider J^k as a finite dimensional vector space over $\mathbf{R}$ and $\mathfrak{R}_0^k$ (respectively, $\mathfrak{R}_l^k$, $l = 1, \ldots, k$) as an open subset (respectively, an affine subspace) of J^k. It is well known that $\mathfrak{R}_l^k$ has the structure of a Lie group, and as such it acts smoothly on the left on J^k, by the formula $j^k(h) \cdot j^k(f) = j^k(f \circ h^{-1})$. Hence, any orbit of $\mathfrak{R}_l^k$ in J^k is a submanifold in J^k. For $f \in \mathfrak{M}$, denote by $V_l^k(f)$ the orbit of $\mathfrak{R}_l^k$ in J^k passing through $j^k(f)$; codim $V_l^k(f)$ is the codimension of $V_l^k(f)$ in J^k.

Define $\pi\colon J^{k+1} \ni z \to j^k(z) \in J^k$. For $f \in \mathfrak{M}$ and $0 \leqslant s \leqslant q$, let

$$A_s^q(f)\colon (\mathfrak{M}^{s+1}/\mathfrak{M}^{q+1})^n \to \mathfrak{M}^{s+1}/\mathfrak{M}^{q+1}$$

be the $\mathbf{R}$-linear mapping defined by the formula

$$A_s^q(f)(\bar{u}_1, \ldots, \bar{u}_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \bar{u}_i,$$

where $\bar{u}$ denotes the equivalence class of $u \in \mathfrak{M}$ modulo $\mathfrak{M}^{q+1}$. Note that $j^{q-s}(f) = j^{q-s}(g)$ implies $A_s^q(f) = A_s^q(g)$.

THEOREM 1. *Let $w \in J^k$ and $1 \leqslant l \leqslant k$. The following conditions are equivalent*:

(1) *The k-jet w is $\mathfrak{R}_l$ sufficient*;
(2) $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(w)$;
(2′) $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(f)$ *for each* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(2″) $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1} I(f)$ *for some* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(3) $\binom{n+k}{k+1} + \operatorname{rank} A_l^k(w) = \operatorname{rank} A_l^{k+1}(w)$;
(3′) $\binom{n+k}{k+1} + \operatorname{rank} A_l^k(f) = \operatorname{rank} A_l^{k+1}(f)$ *for each* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(3″) $\binom{n+k}{k+1} + \operatorname{rank} A_l^k(f) = \operatorname{rank} A_l^{k+1}(f)$ *for some* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(4) codim $V_l^k(w) =$ codim $V_l^{k+1}(w)$;
(4′) codim $V_l^k(f) =$ codim $V_l^{k+1}(f)$ *for each* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(4″) codim $V_l^k(f) =$ codim $V_l^{k+1}(f)$ *for some* $f \in \mathfrak{M}$ *with* $j^k(f) = w$.

COROLLARY 1. *For $f \in \mathfrak{M}$ and $1 \leqslant l \leqslant k$ the following conditions are equivalent*:

(i) $j^k(f)$ *is $\mathfrak{R}_l$ sufficient*;
(ii) $\forall s > k$, codim $V_l^k(f) =$ codim $V_l^s(f)$;
(iii) $\exists s > k \ni$ codim $V_l^k(f) =$ codim $V_l^s(f)$.

THEOREM 2. *For a given k-jet $w \in J^k$ the following conditions are equivalent*:

(a) *The k-jet w is sufficient*;
(b) $\mathfrak{M}^{k+1} \subset \mathfrak{M} I(z)$ *for each* $z \in \pi^{-1}(w)$;
(b′) $\mathfrak{M}^{k+1} \subset \mathfrak{M} I(f)$ *for each* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(c) $\binom{n+k}{k+1} + \operatorname{rank} A_0^k(z) = \operatorname{rank} A_0^{k+1}(z)$ *for each* $z \in \pi^{-1}(w)$;
(c′) $\binom{n+k}{k+1} + \operatorname{rank} A_0^k(z) = \operatorname{rank} A_0^{k+1}(z)$ *for each* $f \in \mathfrak{M}$ *with* $j^k(f) = w$;
(d) codim $V_0^k(z) =$ codim $V_0^{k+1}(z)$ *for each* $z \in \pi^{-1}(w)$;
(d′) codim $V_0^k(f) =$ codim $V_0^{k+1}(f)$ *for each* $f \in \mathfrak{M}$ *with* $j^k(f) = w$.

From this we have

COROLLARY 2. *If a k-jet $j^k(f)$ of $f \in \mathfrak{M}$ is sufficient, then* codim $V_0^s(f)$ = codim $V_0^k(f)$ *for each* $s \geqslant k$.

Observe that, given $1 \leqslant l \leqslant k$ and $f, g \in \mathfrak{M}$ with $j^k(f) = j^k(g)$, we have $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1}I(f)$ if and only if $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1}I(g)$ (by Nakayama's lemma). If $l = 0$, the last equivalence does not hold (see Theorem 2 and Counterexample 1).

THEOREM 3. *The set Ω_l^k of $\mathfrak{R}_l$ sufficient k-jets is a semialgebraic subset of J^k, $0 \leqslant l \leqslant k$.*

One can estimate the codimension of Ω_l^k in J^k. Given $k, l, p \in \mathbf{N}$, $k \geqslant p$, put $\omega(k,l) = \sup\{r \in \mathbf{N}: k \geqslant (r+1)^n + l + 1\}$, and denote by $\pi_{k,p}: J^k \to J^p$ the natural linear projection of J^k onto J^p.

PROPOSITION 1. codim $\Omega_l^k \leqslant$ dim ker $\pi_{k,\omega(k,l)}$ *for* $l = 1, \ldots, k$ *and* codim $\Omega_0^k \leqslant$ codim Ω_1^k.

3. **Proofs.** Put $\pi^k: \mathfrak{M} \ni g \to j^k(g) \in J^k$. We need the following:

LEMMA 1 [3]. *Let $f \in \mathfrak{M}$ and $0 \leqslant l \leqslant k$. The linear subspace $\pi^k(\mathfrak{M}^{l+1}I(f))$ is the tangent space of $V_l^k(f)$ at $j^k(f)$.*

LEMMA 2. *For given $f \in \mathfrak{M}$ and $0 \leqslant l \leqslant k$, the following conditions are equivalent*:

(i) $\mathfrak{M}^{k+1} \subset \mathfrak{M}^{l+1}I(f)$;

(ii) rank $A_l^{k+1}(f) = \binom{n+k}{k+1} +$ rank $A_l^k(f)$;

(iii) codim $V_l^{k+1}(f) =$ codim $V_l^k(f)$.

PROOF. Observe that, by Nakayama's lemma, (i) is equivalent to

$$(*) \qquad \mathfrak{M}^{k+1} + \mathfrak{M}^{l+1}I(f) = \mathfrak{M}^{k+2} + \mathfrak{M}^{l+1}I(f).$$

Condition $(*)$ is trivially equivalent to

$$(**) \quad \dim_{\mathbf{R}}(\mathfrak{M}^{k+1} + \mathfrak{M}^{l+1}I(f)/\mathfrak{M}^{k+2}) = \dim_{\mathbf{R}}(\mathfrak{M}^{k+2} + \mathfrak{M}^{l+1}I(f)/\mathfrak{M}^{k+2}).$$

The well-known fact that $\mathbf{R}$-linear spaces $(\mathfrak{M}^{k+1} + \mathfrak{M}^{l+1}I(f)/\mathfrak{M}^{k+1})$ and $(\mathfrak{M}^{k+1} + \mathfrak{M}^{l+1}I(f)/\mathfrak{M}^{k+2})/(\mathfrak{M}^{k+1}/\mathfrak{M}^{k+2})$ are isomorphs, and $(**)$ imply (i) $\Leftrightarrow$ (ii), because $(\mathfrak{M}^{q+1} + \mathfrak{M}^{l+1}I(f)/\mathfrak{M}^{q+1})$ is the image of $A_l^q(f)$ and $\dim_{\mathbf{R}} \mathfrak{M}^{k+1}/\mathfrak{M}^{k+2} = \binom{n+k}{k+1}$.

Lemma 1 implies that $\mathbf{R}$-linear spaces $(\mathfrak{M}/\mathfrak{M}^{k+1} + \mathfrak{M}^{l+1}I(f))$ and $J^k/T_z V_l^k(f)$ are isomorphs, $z = j^k(f)$. It is clear that $(*)$ is equivalent to

$$(***) \qquad \dim_{\mathbf{R}}(\mathfrak{M}/\mathfrak{M}^{k+1} + \mathfrak{M}^{l+1}I(f)) = \dim_{\mathbf{R}}(\mathfrak{M}/\mathfrak{M}^{k+2} + \mathfrak{M}^{l+1}I(f)).$$

Combining these facts one obtains (i) $\Leftrightarrow$ (iii).

PROOF OF THEOREM 1. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2′) $\Leftrightarrow$ (2″) follows from Theorems I and II. (3) $\Leftrightarrow$ (3′) $\Leftrightarrow$ (3″) is an immediate consequence of the definition of $A_l^k(f)$. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from Lemma 2.

(4) $\Leftrightarrow$ (4′) $\Leftrightarrow$ (4″). The fact that $j^k(w) = j^k(f)$ implies $\mathfrak{M}^{k+s+1} + \mathfrak{M}^{l+1}I(w) = \mathfrak{M}^{k+s+1} + \mathfrak{M}^{l+1}I(f)$, $s = 0, 1$, because $l \geqslant 1$. Equalities codim $V_l^{k+s}(f)$ = codim $V_l^{k+s}(w)$, $s = 0, 1$, $l \geqslant 1$, follow from the last observation, since $\mathbf{R}$-

linear spaces $(\mathfrak{M}/\mathfrak{M}^{k+s+1} + \mathfrak{M}^{l+1}I(f))$ and $J^{k+s}/T_z V_l^{k+s}(f)$ are isomorphs, $f \in \mathfrak{M}$, $z = j^{k+s}(f)$, $s = 0, 1$, $l \geqslant 1$.

PROOF OF COROLLARY 1. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) follow by induction from Theorem 1(observe that if $j^k(f)$ is $\mathfrak{R}_l$ sufficient, then $j^q(f)$ is also $\mathfrak{R}_l$ sufficient for $q \geqslant k$). Implication (ii) $\Rightarrow$ (iii) is trivial.

PROOF OF THEOREM 2. (a) $\Rightarrow$ (b) and (b) $\Leftrightarrow$ (b′) follow, respectively, from Theorem II and Nakayama's lemma.

(b) $\Rightarrow$ (a). First recall the following theorem of Mather [3, IV]: Let C^∞ Lie group G act on C^∞ manifolds M and N. Suppose that C^∞ mapping F: $M \to N$ is G-submersion (i.e. F is a submersion and $F(xg) = F(x)g$ for $x \in M$, $g \in G$). Let $y \in N$ and $V = F^{-1}(y)$. If V is a connected submanifold of M, then V is contained in a single orbit of the group G in M if and only if $T_x V \subset T_x(xG)$ for each $x \in V$.

We shall use this theorem in the following situation. The group $\mathfrak{R}_0^{k+1}$ acts on J^{k+1}. Define the action of $\mathfrak{R}_0^{k+1}$ on J^k by $J^k \times \mathfrak{R}_0^{k+1} \ni (z, h) \to j^k(z \circ h^{-1}) \in J^k$. The mapping π: $J^{k+1} \ni z \to j^k(z) \in J^k$ is a $\mathfrak{R}_0^{k+1}$-submersion, and the set $U^{k+1} = \pi^{-1}(w)$, as an affine subspace of J^{k+1}, is a connected submanifold of J^{k+1}. Consider w as an element of J^{k+1}, $w \in U^{k+1}$. The tangent space of U^{k+1} at w is the space $\pi^{k+1}(\mathfrak{M}^{k+1})$, and by Lemma 1 the tangent space of $V_0^{k+1}(w)$ at w equals $\pi^{k+1}(\mathfrak{M}I(w))$. By condition (b) we have $\mathfrak{M}^{k+1} \subset \mathfrak{M}I(w)$ and, hence, also, $\pi^{k+1}(\mathfrak{M}^{k+1}) \subset \pi^{k+1}(\mathfrak{M}I(w))$. Observe that $w \in V_0^{k+1}(w) \cap U^{k+1}$ and, hence, by Mather's theorem U^{k+1} is contained in $V_0^{k+1}(w)$. Now we can prove that $w \in J^k$ is a sufficient jet. Take $f \in \mathfrak{M}$ with $j^k(f) = w$. There exists a germ $h \in \mathfrak{R}_0$ such that $j^{k+1}(f) = j^{k+1}(w \circ h)$, because $j^{k+1}(f) \in U^{k+1} \subset V_0^{k+1}(w)$. (b) implies $\mathfrak{M}^{k+2} \subset \mathfrak{M}^2 I(w \circ h)$, which is equivalent to the $\mathfrak{R}_1$ sufficiency of $j^{k+1}(w \circ h) \in J^{k+1}$. Thus there exists a germ $h' \in \mathfrak{R}_0$ such that $f = w \circ h \circ h'$, which proves the sufficiency of k-jet w.

(b) $\Leftrightarrow$ (c) and (b) $\Leftrightarrow$ (d) follow from Lemma 2. (c) $\Leftrightarrow$ (c′) is, as earlier, an easy consequence of the definition of $A_0^k(z)$. A proof of (d) $\Leftrightarrow$ (d′) is similar to one for Theorem 1, (4) $\Leftrightarrow$ (4′).

Corollary 2 can be deduced from Theorem 2 by simple induction on s.

PROOF OF THEOREM 3. Let $1 \leqslant l \leqslant k$. By Theorem 1 we have

$$\Omega_l^k = \left\{ w \in J^k : \binom{n+k}{k+1} + \operatorname{rank} A_l^k(w) = \operatorname{rank} A_l^{k+1}(w) \right\}.$$

Hence Ω_l^k is a semialgebraic subset of J^k, $1 \leqslant l \leqslant k$. Now put

$$\Delta_0^{k+1} = \{ z \in J^{k+1} : \mathfrak{M}^{k+1} \not\subset \mathfrak{M}I(z) \}.$$

The set Δ_0^{k+1} is a semialgebraic subset of J^{k+1} because, by Lemma 2,

$$\Delta_0^{k+1} = \left\{ z \in J^{k+1} : \binom{n+k}{k+1} + \operatorname{rank} A_0^k(z) > \operatorname{rank} A_0^{k+1}(z) \right\}.$$

Theorem 2 implies that $\pi(\Delta_0^{k+1}) = J^k \backslash \Omega_0^k$. Hence by the Seidenberg-Tarski theorem [5], the set $J^k \backslash \Omega_0^k$ – as a linear projection of a semialgebraic set–is semialgebraic. Clearly Ω_0^k is also a semialgebraic set and Theorem 3 is proved.

PROOF OF PROPOSITION 1. Inequality codim $\Omega_0^k \leqslant$ codim Ω_1^k is obvious. Let

$1 \leqslant l \leqslant k$. Fix $k \in N$ and put $q = (k+1)^n + l + 1$. Observe that for $w \in J^k$ we have $\pi_{q,k}^{-1}(w) \cap \Omega_l^q \neq \varnothing$. If $0 \in \mathbf{R}^n$ is a regular point of w, this is trivial. In the other case, if 0 is a critical point of w, we can choose a homogeneous polynomial h of degree $k+1$ such that $0 \in \mathbf{C}^n$ is an isolated critical point of $w + h$, where $\widetilde{w+h}$ is the complexification of $\widetilde{w+h}$. A theorem of Samuel [2] implies $\mathfrak{M}^{(k+1)^n} \subset I(w+h)$, and hence $j^q(w+h) \in \pi_{q,h}^{-1}(w) \cap \Omega_l^q$ (Theorem I). This means that $\pi_{k,\omega(k,l)}(\Omega_l^k) = J^{\omega(k,l)}$ and, hence, $\operatorname{codim} \Omega_l^k \leqslant \dim \ker \pi_{k,\omega(k,l)}$.

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