A CHARACTERIZATION OF $C^\infty$-SUFFICIENT k-JETS

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Abstract. We improve some results of Mather and Arnold and find several necessary and sufficient conditions of sufficiency of k-jets. As a corollary we prove that the set of $C^\infty$-sufficient k-jets is a semialgebraic subset of the space of k-jets of $C^\infty$ mappings $F: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$.

1. Introduction. Let $\mathcal{E}$ denote the local $\mathbb{R}$-algebra of germs at $0 \in \mathbb{R}^n$ of all $C^\infty$ functions from $\mathbb{R}^n$ to $\mathbb{R}$, and let $\mathfrak{M}$ be the unique maximal ideal of $\mathcal{E}$. For a given $f \in \mathcal{E}$, let $I(f)$ denote the ideal in $\mathcal{E}$ generated by $\partial f / \partial x_1, \ldots, \partial f / \partial x_n$. The space of k-jets of $C^\infty$ mappings $h: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ is denoted by $J_k(n, p)$ and, by definition, $J_k(n, 1) = J_k(f)$ is the k-jet of $f$. Let $\mathfrak{G}_k$ denote the group of germs of $C^\infty$ diffeomorphisms $h: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$, such that $j^k(h)$ coincides with the k-jet of the identity mapping $\mathbb{R}^n$ onto itself. Clearly $\mathfrak{G}_0$ is the set of all local diffeomorphisms around 0 in $\mathbb{R}^n$.

Definition. A k-jet $w \in J_k$ is called $\mathfrak{G}_l$ sufficient ($0 \leq l \leq k$) if for any $f \in \mathcal{E}$ such that $j^k(f) = w$, there exists $h \in \mathfrak{G}_l$ such that $f \circ h = w$. A $\mathfrak{G}_0$ sufficient jet is called sufficient.

The following theorems are known:

Theorem I (Mather [4], Arnold [1]). Let $f \in \mathfrak{M}$ and $1 \leq l \leq k$. Suppose $\mathfrak{M}^{k+1} \subseteq \mathfrak{M}^{l+1} I(f)$. Then the k-jet $j^k(f)$, of $f$ is $\mathfrak{G}_l$ sufficient.

Theorem II (Mather [3]). Let $f \in \mathfrak{M}$ and $0 \leq l \leq k$. If the k-jet $j^k(f)$ is $\mathfrak{G}_l$ sufficient, then $\mathfrak{M}^{k+1} \subseteq \mathfrak{M}^{l+1} I(f)$.

In particular for $1 \leq l \leq k$, a k-jet $w \in J_k$ is $\mathfrak{G}_l$ sufficient if and only if $\mathfrak{M}^{k+1} \subseteq \mathfrak{M}^{l+1} I(w)$. As a corollary one can observe that for $w \in J_k$, the condition $\mathfrak{M}^{k+1} \subseteq \mathfrak{M}^2 I(w)$ implies the sufficiency of $w$; and if $w$ is sufficient, then $\mathfrak{M}^{k+1} \subseteq \mathfrak{M} I(w)$. However, none of these conditions characterizes the sufficiency of k-jets.

Counterexample 1 (communicated by J. Robbin). For $w = x^2 + 2xy^2 \in J^3$ we have $\mathfrak{M}^4 \subseteq \mathfrak{M} I(w)$, but the jet $w$ is not sufficient. In fact, $0 \in \mathbb{R}^2$ is an isolated critical point for $w$, but not for $w + y^4$.

Counterexample 2. Let $w(x, y) = x^3 + xy^3 \in J^4$. Arnold proved [1, p. 12] that the jet $w$ is sufficient (it also follows from our Theorem 2). However, one can verify that $y^5 \not\in \mathfrak{M}^2 I(w)$ and, hence, $\mathfrak{M}^5 \not\subseteq \mathfrak{M}^2 I(w)$.

Here we shall improve the results above, and find necessary and sufficient

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conditions of sufficiency of k-jets (Theorem 2). We show also that the set of sufficient k-jets is a semialgebraic subset of J^k (Theorem 3).

2. Results. Put \( \mathcal{R}^k \) = \{ \( J^k(h) \in J^k(n, n): h \in \mathcal{R}_l \), \( l = 0, \ldots, k \). It is clear that one can consider \( J^k \) as a finite dimensional vector space over \( \mathbb{R} \) and \( \mathcal{R}^k_0 \) (respectively, \( \mathcal{R}^k_0 \), \( l = 1, \ldots, k \)) as an open subset (respectively, an affine subspace) of \( J^k \). It is well known that \( \mathcal{R}^k \) has the structure of a Lie group, and as such it acts smoothly on the left on \( J^k \), by the formula \( j^k(h) \cdot j^k(f) = j^k(f \circ h^{-1}) \). Hence, any orbit of \( \mathcal{R}^k \) in \( J^k \) is a submanifold in \( J^k \). For \( f \in \mathcal{M} \), denote by \( V^k(f) \) the orbit of \( \mathcal{R}^k \) in \( J^k \) passing through \( j^k(f) \); codim \( V^k(f) \) is the codimension of \( V^k(f) \) in \( J^k \).

Define \( \pi: J^{k+1} \ni z \rightarrow j^k(z) \in J^k \). For \( f \in \mathcal{M} \) and \( 0 \leq s \leq q \), let

\[
A^{s}_z(f): (\mathcal{M}^{s+1}/\mathcal{M}^{q+1})^n \rightarrow \mathcal{M}^{s+1}/\mathcal{M}^{q+1}
\]

be the \( \mathbb{R} \)-linear mapping defined by the formula

\[
A^{s}_z(f)(\bar{u}_1, \ldots, \bar{u}_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \bar{u}_i,
\]

where \( \bar{u} \) denotes the equivalence class of \( u \in \mathcal{M} \) modulo \( \mathcal{M}^{q+1} \). Note that \( j^{q-s}(f) = j^{q-s}(g) \) implies \( A^{s}_z(f) = A^{s}_z(g) \).

**Theorem 1.** Let \( w \in J^k \) and \( 1 \leq l \leq k \). The following conditions are equivalent:

1. The k-jet \( w \) is \( \mathcal{R}_l \) sufficient;
2. \( \mathcal{M}^{k+1} \subset \mathcal{M}^{l+1} \pi(w) \);
3. \( \mathcal{M}^{k+1} \subset \mathcal{M}^{l+1} \pi(f) \) for each \( f \in \mathcal{M} \) with \( j^k(f) = w \);
4. \( \pi^{k+1} \subset \pi^{l+1}(f) \) for some \( f \in \mathcal{M} \) with \( j^k(f) = w \);
5. \( \pi^{k+1} + \text{rank} A^k_0(w) = \text{rank} A^k_0(w) \);
6. \( \pi^{k+1} + \text{rank} A^k_0(f) = \text{rank} A^k_0(f) \) for each \( f \in \mathcal{M} \) with \( j^k(f) = w \);
7. \( \pi^{k+1} + \text{rank} A^k_0(f) = \text{rank} A^k_0(f) \) for some \( f \in \mathcal{M} \) with \( j^k(f) = w \);
8. \( \text{codim} V^k_0(w) = \text{codim} V^k_0(f) \);
9. \( \text{codim} V^k_0(f) = \text{codim} V^k_0(f) \) for each \( f \in \mathcal{M} \) with \( j^k(f) = w \);
10. \( \text{codim} V^k_0(f) = \text{codim} V^k_0(f) \) for some \( f \in \mathcal{M} \) with \( j^k(f) = w \).

**Corollary 1.** For \( f \in \mathcal{M} \) and \( 1 \leq l \leq k \) the following conditions are equivalent:

1. \( j^k(f) \) is \( \mathcal{R}_l \) sufficient;
2. \( \forall s \geq k, \text{codim} V^{k}_l(f) = \text{codim} V^{k}_s(f) \);
3. \( \exists s \geq k \Rightarrow \text{codim} V^{k}_l(f) = \text{codim} V^{k}_s(f) \).

**Theorem 2.** For a given k-jet \( w \in J^k \) the following conditions are equivalent:

1. The k-jet \( w \) is sufficient;
2. \( \mathcal{M}^{k+1} \subset \mathcal{M} \pi(z) \) for each \( z \in \pi^{-1}(w) \);
3. \( \mathcal{M}^{k+1} \subset \mathcal{M} \pi(f) \) for each \( f \in \mathcal{M} \) with \( j^k(f) = w \);
4. \( \pi^{k+1} + \text{rank} A^k_0(z) = \text{rank} A^k_0(z) \) for each \( z \in \pi^{-1}(w) \);
5. \( \pi^{k+1} + \text{rank} A^k_0(z) = \text{rank} A^k_0(z) \) for each \( f \in \mathcal{M} \) with \( j^k(f) = w \);
6. \( \text{codim} V^k_0(z) = \text{codim} V^k_0(z) \) for each \( z \in \pi^{-1}(w) \);
7. \( \text{codim} V^k_0(f) = \text{codim} V^k_0(f) \) for each \( f \in \mathcal{M} \) with \( j^k(f) = w \).

From this we have...
Corollary 2. If a $k$-jet $j^k(f)$ of $f \in \mathcal{M}$ is sufficient, then $\text{codim } \mathcal{V}_0^s(f) = \text{codim } \mathcal{V}_0^k(f)$ for each $s \geq k$.

Observe that, given $1 \leq l \leq k$ and $f, g \in \mathcal{M}$ with $j^k(f) = j^k(g)$, we have $\mathcal{M}^{k+1} \subset \mathcal{M}^{l+1} I(f)$ if and only if $\mathcal{M}^{k+1} \subset \mathcal{M}^{l+1} I(g)$ (by Nakayama's lemma). If $l = 0$, the last equivalence does not hold (see Theorem 2 and Counterexample 1).

Theorem 3. The set $\Omega^k_l$ of $\mathcal{R}_i$ sufficient $k$-jets is a semialgebraic subset of $J^k$, $0 \leq l \leq k$.

One can estimate the codimension of $\Omega^k_l$ in $J^k$. Given $k, l, p \in \mathbb{N}$, $k \geq p$, put $\omega(k, l) = \sup(r \in \mathbb{N}: k \geq (r + l)^n + l + 1)$, and denote by $\pi_{k, p} : J^k \to J^p$ the natural linear projection of $J^k$ onto $J^p$.

Proposition 1. $\text{codim } \Omega^k_l \leq \dim \ker \pi_{k, \omega(k, l)}$ for $l = 1, \ldots, k$ and $\text{codim } \Omega^k_0 \leq \text{codim } \Omega^k_k$.

3. Proofs. Put $\pi^k : \mathcal{M} \ni g \to j^k(g) \in J^k$. We need the following:

Lemma 1 [3]. Let $f \in \mathcal{M}$ and $0 \leq l \leq k$. The linear subspace $\pi^k(\mathcal{M}^{l+1} I(f))$ is the tangent space of $V^k_l(f)$ at $j^k(f)$.

Lemma 2. For given $f \in \mathcal{M}$ and $0 \leq l \leq k$, the following conditions are equivalent:

(i) $\mathcal{M}^{k+1} \subset \mathcal{M}^{l+1} I(f)$;
(ii) $\text{rank } A^k_{l+1} (f) = (\frac{l+k}{k+1}) + \text{rank } A^k_{l} (f)$;
(iii) $\text{codim } \mathcal{V}^k_{l+1} (f) = \text{codim } \mathcal{V}^k_{l} (f)$.

Proof. Observe that, by Nakayama’s lemma, (i) is equivalent to

\[ (*) \quad \mathcal{M}^{k+1} + \mathcal{M}^{l+1} I(f) = \mathcal{M}^{k+2} + \mathcal{M}^{l+1} I(f). \]

Condition $(*)$ is trivially equivalent to

\[ (**) \quad \dim_{\mathbb{R}} (\mathcal{M}^{k+1} + \mathcal{M}^{l+1} I(f)/\mathcal{M}^{k+2}) = \dim_{\mathbb{R}} (\mathcal{M}^{k+2} + \mathcal{M}^{l+1} I(f)/\mathcal{M}^{k+2}). \]

The well-known fact that $\mathbb{R}$-linear spaces $(\mathcal{M}^{k+1} + \mathcal{M}^{l+1} I(f)/\mathcal{M}^{k+1})$ and $(\mathcal{M}^{k+1} + \mathcal{M}^{l+1} I(f)/\mathcal{M}^{k+2})/(\mathcal{M}^{k+1}/\mathcal{M}^{k+2})$ are isomorphs, and $(**)$ imply (i) $\iff$ (ii), because $(\mathcal{M}^{l+1} + \mathcal{M}^{l+1} I(f)/\mathcal{M}^{l+1})$ is the image of $A^k_{l} (f)$ and $\dim_{\mathbb{R}} \mathcal{M}^{k+1}/\mathcal{M}^{k+2} = (\frac{k+l}{k+1})$.

Lemma 1 implies that $\mathbb{R}$-linear spaces $(\mathcal{M}/\mathcal{M}^{k+1} + \mathcal{M}^{l+1} I(f))$ and $J^k/\mathcal{V}^k_{l} (f)$ are isomorphs, $z = j^k(f)$. It is clear that $(*)$ is equivalent to

\[ (***) \quad \dim_{\mathbb{R}} (\mathcal{M}/\mathcal{M}^{k+1} + \mathcal{M}^{l+1} I(f)) = \dim_{\mathbb{R}} (\mathcal{M}/\mathcal{M}^{k+2} + \mathcal{M}^{l+1} I(f)). \]

Combining these facts one obtains (i) $\iff$ (iii).

Proof of Theorem 1. (1) $\iff$ (2) $\iff$ (2') $\iff$ (2'') follows from Theorems I and II. (3) $\iff$ (3') $\iff$ (3'') is an immediate consequence of the definition of $A^k_{l} (f)$.

(2') $\iff$ (3) $\iff$ (4) follows from Lemma 2.

(4) $\iff$ (4') $\iff$ (4''). The fact that $j^k(w) = j^k(f)$ implies $\mathcal{M}^{k+s+1} + \mathcal{M}^{l+1} I(w) = \mathcal{M}^{k+s+1} + \mathcal{M}^{l+1} I(f)$, $s = 0, 1$, because $l \geq 1$. Equalities $\text{codim } \mathcal{V}^k_{l+s} (f) = \text{codim } \mathcal{V}^k_{l+s} (w)$, $s = 0, 1$, $l \geq 1$, follow from the last observation, since $\mathbb{R}$-
linear spaces \((\mathfrak{M}/\mathfrak{M}^{k+s+1} + \mathfrak{M}^{l+1} I(f))\) and \(J^{k+s}/T_z V^{k+s}(f)\) are isomorphs, \(f \in \mathfrak{M}, z = j^{k+s}(f), s = 0, 1, l \geq 1.\)

**Proof of Corollary 1.** (i) \(\Rightarrow\) (ii) and (iii) \(\Rightarrow\) (i) follow by induction from Theorem 1 (observe that if \(j^q(f)\) is \(R_q\) sufficient, then \(j^q(f)\) is also \(R_q\) sufficient for \(q \geq k\)). Implication (ii) \(\Rightarrow\) (iii) is trivial.

**Proof of Theorem 2.** (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (b') follow, respectively, from Theorem II and Nakayama's lemma.

(b) \(\Rightarrow\) (a). First recall the following theorem of Mather [3, IV]: Let \(C^\infty\) Lie group \(G\) act on \(C^\infty\) manifolds \(M\) and \(N\). Suppose that \(C^\infty\) mapping \(F: M \rightarrow N\) is \(G\)-submersion (i.e. \(F\) is a submersion and \(F(xg) = F(x)g\) for \(x \in M, g \in G\)). Let \(y \in N\) and \(V = F^{-1}(y)\). If \(V\) is a connected submanifold of \(M\), then \(V\) is contained in a single orbit of the group \(G\) in \(M\) if and only if \(T_x V \subset T_x(xG)\) for each \(x \in V\).

We shall use this theorem in the following situation. The group \(R_{k+1}\) acts on \(J^{k+1}\). Define the action of \(R_{k+1}\) on \(J^k\) by \(J^k \times R_{k+1} \ni (z, h) \rightarrow \pi(z \circ h^{-1}) \in \mathfrak{M}^l\). The mapping \(\pi: J^{k+1} \ni z \rightarrow j^k(z) \in \mathfrak{M}^k\) is a \(R_{k+1}\)-submersion, and the set \(U^{k+1} = \pi^{-1}(w)\), as an affine subspace of \(J^{k+1}\), is a connected submanifold of \(J^{k+1}\). Consider \(w\) as an element of \(J^{k+1}\), \(w \in U^{k+1}\). The tangent space of \(U^{k+1}\) at \(w\) is the space \(\pi^{k+1}(\mathfrak{M}^{k+1})\), and by Lemma 1 the tangent space of \(V^{k+1}(w)\) at \(w\) equals \(\pi^{k+1}(\mathfrak{M}^l I(w))\). By condition (b) we have \(\mathfrak{M}^{k+1} \subset \mathfrak{M}^l I(w)\) and, hence, also, \(\pi^{k+1}(\mathfrak{M}^{k+1}) \subset \mathfrak{M}^l I(w)\). Observe that \(w \in V^{k+1}(w) \cap U^{k+1}\) and, hence, by Mather's theorem \(U^{k+1}(w)\) is contained in \(V^{k+1}(w)\). Now we can prove that \(w \in J^k\) is a sufficient jet. Take \(f \in \mathfrak{M}\) with \(f^k(f) = w\). There exists a germ \(h \in \mathfrak{M}_0\) such that \(j^{k+1}(f) = j^{k+1}(w \circ h)\), because \(j^{k+1}(f) \in U^{k+1} \subset V^{k+1}(w)\). (b) implies \(\mathfrak{M}^{k+2} \subset \mathfrak{M}^l I(w \circ h)\), which is equivalent to the \(R_k\) sufficiency of \(j^{k+1}(w \circ h) \in J^{k+1}\). Thus there exists a germ \(h' \in \mathfrak{M}_0\) such that \(f = w \circ h \circ h'\), which proves the sufficiency of \(k\)-jet \(w\).

(b) \(\Leftrightarrow\) (c) and (b) \(\Leftrightarrow\) (d) follow from Lemma 2. (c) \(\Rightarrow\) (c') is, as earlier, an easy consequence of the definition of \(A^k_0(z)\). A proof of (d) \(\Leftrightarrow\) (d') is similar to one for Theorem 1, (4) \(\iff\) (4').

Corollary 2 can be deduced from Theorem 2 by simple induction on \(s\).

**Proof of Theorem 3.** Let \(1 \leq l \leq k\). By Theorem 1 we have

\[
\Omega^k_l = \left\{ w \in J^k: \left(\frac{n + k}{k + 1}\right) + \text{rank } A^k_l(w) = \text{rank } A^{k+1}_l(w) \right\}.
\]

Hence \(\Omega^k_l\) is a semialgebraic subset of \(J^k, 1 \leq l \leq k\). Now put

\[
\Delta^k_{l+1} = \left\{ z \in J^{k+1}: \mathfrak{M}^{k+1} \subset \mathfrak{M}^l I(z) \right\}.
\]

The set \(\Delta^k_{l+1}\) is a semialgebraic subset of \(J^{k+1}\) because, by Lemma 2,

\[
\Delta^k_{l+1} = \left\{ z \in J^{k+1}: \left(\frac{n + k}{k + 1}\right) + \text{rank } A^k_l(z) > \text{rank } A^{k+1}_l(z) \right\}.
\]

Theorem 2 implies that \(\pi(\Delta^k_{l+1}) = J^k \setminus \Omega^k_0\). Hence by the Seidenberg-Tarski theorem [5], the set \(J^k \setminus \Omega^k_0 - \) as a linear projection of a semialgebraic set—is semialgebraic. Clearly \(\Omega^k_0\) is also a semialgebraic set and Theorem 3 is proved.

**Proof of Proposition 1.** Inequality \(\text{codim } \Omega^k_0 \leq \text{codim } \Omega^k\) is obvious. Let
$1 \leq l \leq k$. Fix $k \in \mathbb{N}$ and put $q = (k + 1)^n + l + 1$. Observe that for $w \in \mathcal{J}_k^q$ we have $\pi^{-1}_{q,k}(w) \cap \Omega^q_l \neq \emptyset$. If $0 \in \mathbb{R}^n$ is a regular point of $w$, this is trivial. In the other case, if $0$ is a critical point of $w$, we can choose a homogeneous polynomial $h$ of degree $k + 1$ such that $0 \in \mathbb{C}^n$ is an isolated critical point of $w + h$, where $w + h$ is the complexification of $w + h$. A theorem of Samuel [2] implies $\mathfrak{m}_{q,k}(w + h) \subseteq \mathfrak{m}_{q,k}(w) \cap \Omega^q_l$ (Theorem 1). This means that $\pi_{q,k}(\mathcal{J}(\mathcal{J}_k^q)) = \mathcal{J}_k^q$ and,\[ \text{hence, codim } \Omega^q_l \leq \dim \ker \pi_{q,k}(\mathcal{J}_k^q). \]

References

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