A REMARK ON NELSON’S BEST HYPERCONTRACTIVE ESTIMATES

BARRY SIMON

ABSTRACT. By using a combinatorial estimate we provide a new proof of
Nelson’s best hypercontractive estimates from $L^2$ to $L^q$.

Let $G$ be the differential operator
\[- \frac{1}{2} \frac{d}{dx^2} + x \frac{d}{dx} \text{ on } L^2(\mathbb{R}, \pi^{-1/2}e^{-x^2}dx).\]

Hypercontractive estimates on $e^{-tG}$ have played a key role in constructive
quantum field theory; see e.g. [6]. In [5] Nelson proved the estimate
\[\|e^{-tG}f\|_p \leq \|f\|_q\]
if
\[e^{-t} \leq \sqrt{(q-1)/(p-1)}\]
where $\|\cdot\|_p$ is the $L^p(\mathbb{R}, \pi^{-1/2}e^{-x^2}dx)$ norm. (1) is a “best possible” estimate
in the sense that if (2) fails then $e^{-tG}$ is not even bounded from $L^p$ to $L^q$.
Nelson’s proof is quite complicated and the beautiful alternate proof of Gross
[2] involves some computation. Our goal in this note is to give a simple proof
of (1) in case $q = 2; p = \text{even integer}$. This is not the first time that
hypercontractive estimates have been sharper or easier for this case; see the
situation for fermions [3].

Our proof proceeds by a slight strengthening of an argument of Nelson [5]
who easily proves (1) with $p = 4; q = 2$ if $e^{-t} \leq \sqrt{1/4}$. Nelson’s argument
extends to $p = 2k, q = 2$ if $e^{-t} \leq \sqrt{1/2k}$ ($k = \text{integer}$). Let $A_k(n)$ be defined
as follows. Consider $2kn$ objects broken into $2k$ groups of $n$ objects each.
$A_k(n)$ is the number of ways of assigning these $2kn$ objects into $kn$ pairs in
such a way that no two objects in the same group are paired with each other.
Thus e.g.
\[A_1(n) = n!\]

Obviously, $A_k(n)$ is dominated by the total number of pairings without any
restriction and this is $(2kn)!/(kn)!2^{kn}$. From this one finds that
\[A_k(n) \leq (2k)^{kn}(A_1(n))^k.\]
(4) is the basis of the easy Nelson proof mentioned above. By mimicking Nelson’s proof, the best estimates from $L^2$ to $L^{2k}$ follow from the following combinatorial result which is the main result of this note:

**Theorem 1.** $A_k(n) \leq (2k - 1)^k A_1(n)^k$.

**Proof.** We will show that

$$A_{kn} \leq \left(\frac{2k - 1}{2k}\right)^{kn}[2kn(2kn - 2)(2kn - 4) \cdots 2].$$  \hspace{1cm} (5)

The last factor in (5) is $2^{kn}(kn)!$. By the multinomial theorem $(kn)! \leq k^{kn}(n!)^k$ so (5) implies the estimate of the theorem. Let us give an algorithm for finding all allowed pairings and then estimate the number of choices at each stage. Write $2kn$ objects as $a_1^{(1)}, \ldots, a_n^{(1)}; a_1^{(2)} \cdots; \ldots, a_n^{(2k)}$. At each stage choose the group with the most unpaired elements left (if several groups have equal numbers left choose the one with smallest group number $y$ in $a_y^{(j)}$). In the group $a_j^{(i)}$ chosen, pair the $a_i^{(j)}$ with $i$ smallest with some element in some other group. This algorithm will clearly yield each allowed pairing once. After $m$ pairs have been chosen, $2kn - 2m$ elements remain. At least $(2kn - 2m)/2k$ of those elements lie in the group with the most unpaired elements so at the $(m + 1)$st pairing, at most $[(2k - 1)/2k]2^{kn - 2m}$ choices are available. This proves the bound (5).

We would also like to make a remark about the best possible nature of the hypercontractive bounds. For a semigroup $e^{-tG}$ taking 1 into 1, there is a close connection between $G$ having a gap in its spectrum above zero and $e^{-tG}$ being a contraction from $L^2$ to $L^4$ for some $t$. Glimm [1] proved that if $G$ has a gap and if $e^{-tG}$ is bounded from $L^2$ to $L^4$ for some $t_0$, it is a contraction for sufficiently large $t$. Guerra, Rosen and Simon [4] proved that if $e^{-tG}$ generates a Markov process, then $e^{-tG}$ a contraction from $L^2$ to $L^4$ implies a mass gap for $G$. By “running Glimm’s proof backwards”, we can sharpen the GRS result:

**Theorem 2.** Let $T$ be a reality preserving bounded operator on $L^2(M; d\mu)$; $\mu(M) = 1$ so that (a) $T1 = 1$, (b) $T$ is a contraction from $L^2$ to $L^4$. Then, $T^*1 = 1$ and $\|T^\dagger\{1\}\| \leq \sqrt{1/3}$.

**Proof.** Let $f = \alpha 1 + g$ with $g \in \{1\}^\perp$, $\alpha$ real and $g$ real valued. Then

$$\|f\|^4_2 = (\alpha^4 + 2\alpha^2\|g\|^2_2 + \|g\|^4_2)$$

and

$$\|Tf\|^4_4 = \alpha^4 + 4\alpha^3\langle 1, Tg \rangle + 6\alpha^2\|Tg\|^2_2 + O(\alpha).$$

By hypothesis: $\|Tf\|_4 \ll \|f\|_2$ so taking $\alpha$ large we have $\langle 1, Tg \rangle \ll 0$. This implies that $\langle 1, Tg \rangle = 0$ so that $T^*1 = \alpha 1$. Since $\langle 1, T^*1 \rangle = \langle T1, 1 \rangle = 1$, $\alpha = 1$. Since $\langle 1, Tg \rangle = 0$, taking $\alpha$ large we have

$$6\|Tg\|^2_2 \ll 2\|g\|^2_2 \quad \text{or} \quad \|Tg\|_2 \ll \sqrt{1/3}\|g\|_2.$$  \hspace{1cm} \Box

**Remark.** Thus, the best possible estimate from $L^2$ to $L^4$ implies that $G$ has a gap of size 1. If a better estimate held, the gap would be bigger than 1. Since
G has a gap of precisely one, we have the best possible nature of the estimates.

It is a pleasure to thank C. Fefferman for a useful remark.

Note. After the completion of this manuscript I learned of two new proofs of the full best hypercontractive estimates, one by W. Beckner and the other by H. Brascamp and E. Lieb.

REFERENCES