A REMARK ON NELSON’S BEST HYPERCONTRACTIVE ESTIMATES

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Abstract. By using a combinatorial estimate we provide a new proof of Nelson’s best hypercontractive estimates from $L^2$ to $L^4$.

Let $G$ be the differential operator

$$
-\frac{1}{2} \frac{d^2}{dx^2} + x \frac{d}{dx} \quad \text{on} \quad L^2(\mathbb{R}, \pi^{-1/2} e^{-x^2} dx).
$$

Hypercontractive estimates on $e^{-tG}$ have played a key role in constructive quantum field theory; see e.g. [6]. In [5] Nelson proved the estimate

$$
\|e^{-tG}f\|_p \leq \|f\|_q
$$

if

$$
e^{-t} \leq \sqrt{(q-1)/(p-1)}
$$

where $\|\cdot\|_p$ is the $L^p(\mathbb{R}, \pi^{-1/2} e^{-x^2} dx)$ norm. (1) is a “best possible” estimate in the sense that if (2) fails then $e^{-tG}$ is not even bounded from $L^p$ to $L^q$. Nelson’s proof is quite complicated and the beautiful alternate proof of Gross [2] involves some computation. Our goal in this note is to give a simple proof of (1) in case $q=2$; $p$ = even integer. This is not the first time that hypercontractive estimates have been sharper or easier for this case; see the situation for fermions [3].

Our proof proceeds by a slight strengthening of an argument of Nelson [5] who easily proves (1) with $p=4$; $q=2$ if $e^{-t} \leq \sqrt{1/4}$. Nelson’s argument extends to $p=2k$, $q=2$ if $e^{-t} \leq \sqrt{1/2k}$ ($k$ = integer). Let $A_k(n)$ be defined as follows. Consider $2kn$ objects broken into $2k$ groups of $n$ objects each. $A_k(n)$ is the number of ways of assigning these $2kn$ objects into $kn$ pairs in such a way that no two objects in the same group are paired with each other. Thus e.g.

$$
A_1(n) = n!.
$$

Obviously, $A_k(n)$ is dominated by the total number of pairings without any restriction and this is $(2kn)!/(kn)!2^k n$. From this one finds that

$$
A_k(n) \leq (2k)^{kn} (A_1(n))^k.
$$

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(4) is the basis of the easy Nelson proof mentioned above. By mimicking Nelson’s proof, the best estimates from $L^2$ to $L^{2k}$ follow from the following combinatorial result which is the main result of this note:

**Theorem 1.** $A_k(n) \leq (2k - 1)^k A_1(n)^k$.

**Proof.** We will show that

$$ A_{kn} \leq \left[\frac{(2k - 1)}{2k}\right]^{kn} \left[\frac{(2kn)}{(2kn - 2)(2kn - 4) \cdots 2}\right]. $$

The last factor in (5) is $2^{kn}(kn)!$. By the multinomial theorem $(kn)! < k^{kn}(n!)^k$ so (5) implies the estimate of the theorem. Let us give an algorithm for finding all allowed pairings and then estimate the number of choices at each stage. Write $2kn$ objects as $\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}; \alpha_1^{(2)}; \ldots; \alpha_n^{(2k)}$. At each stage choose the group with the most unpaired elements left (if several groups have equal numbers left choose the one with smallest group number $y$ in $\alpha_i^{(y)}$). In the group $\alpha_i^{(y)}$ chosen, pair the $\alpha_i^{(y)}$ with $i$ smallest with some element in some other group. This algorithm will clearly yield each allowed pairing once. After $m$ pairs have been chosen, $2kn - 2m$ elements remain. At least $(2kn - 2m)/2k$ of those elements lie in the group with the most unpaired elements so at the $(m + 1)$st pairing, at most $[(2k - 1)/2k][2kn - 2m]$ choices are available. This proves the bound (5).

We would also like to make a remark about the best possible nature of the hypercontractive bounds. For a semigroup $e^{-tG}$ taking 1 into 1, there is a close connection between $G$ having a gap in its spectrum above zero and $e^{-tG}$ being a contraction from $L^2$ to $L^4$ for some $t$. Glimm [1] proved that if $G$ has a gap and if $e^{-tG}$ is bounded from $L^2$ to $L^4$ for some $t_0$, it is a contraction for sufficiently large $t$. Guerra, Rosen and Simon [4] proved that if $e^{-tG}$ generates a Markov process, then $e^{-tG}$ a contraction from $L^2$ to $L^4$ implies a mass gap for $G$. By “running Glimm’s proof backwards”, we can sharpen the GRS result:

**Theorem 2.** Let $T$ be a reality preserving bounded operator on $L^2(M; d\mu)$; $\mu(M) = 1$ so that (a) $T1 = 1$, (b) $T$ is a contraction from $L^2$ to $L^4$. Then, $T^*1 = 1$ and $\|T \{1\}^\perp\| \leq \sqrt[4]{1/3}$.

**Proof.** Let $f = \alpha 1 + g$ with $g \in \{1\}^\perp$, $\alpha$ real and $g$ real valued. Then

$$ \|f\|_4^4 = (\alpha^4 + 2\alpha^2\|g\|_2^2 + \|g\|_2^4) $$

and

$$ \|Tf\|_4^4 = \alpha^4 + 4\alpha^3\langle 1, Tg \rangle + 6\alpha^2\|Tg\|_2^2 + O(\alpha). $$

By hypothesis: $\|T\|_4 \leq \|f\|_2$ so taking $\alpha$ large we have $\langle 1, Tg \rangle \leq 0$. This implies that $\langle 1, Tg \rangle = 0$ so that $T^*1 = \alpha 1$. Since $\langle 1, T^*1 \rangle = \langle T1, 1 \rangle = 1$, $\alpha = 1$. Since $\langle 1, Tg \rangle = 0$, taking $\alpha$ large we have

$$ 6\|Tg\|_2^2 \leq 2\|g\|_2^2 \quad \text{or} \quad \|Tg\|_2 < \sqrt[6]{1/3} \|g\|_2. $$

**Remark.** Thus, the best possible estimate from $L^2$ to $L^4$ implies that $G$ has a gap of size 1. If a better estimate held, the gap would be bigger than 1. Since
$G$ has a gap of precisely one, we have the best possible nature of the estimates.

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*Note.* After the completion of this manuscript I learned of two new proofs of the full best hypercontractive estimates, one by W. Beckner and the other by H. Brascamp and E. Lieb.

**References**


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