ON PURE STATES OF C*-SUBALGEBRAS

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In [1, 2.12.21, p. 58] the following question is raised. Let $\mathcal{A}$ be a C*-algebra, $\mathcal{B}$ a C*-subalgebra of $\mathcal{A}$ and $q$ a nonzero positive element of $\mathcal{A}$. Does there exist a state $\varphi$ on $\mathcal{B}$ such that the restriction of $\varphi$ to $\mathcal{B}$ is a pure state and $\varphi(q) > 0$? The question was answered in the negative in [2]. Our purpose in this note is to present another proof of this fact.

**Proposition.** Let $\mathcal{A}$ be a C*-algebra and let $q$, $a$ and $s$ be elements of $\mathcal{A}$ such that $q$ is nonzero and positive, $a$ is selfadjoint, and $q = as - sa$. Let $\mathcal{B}$ be an abelian C*-subalgebra of $\mathcal{A}$ which contains $a$ and the identity $e$. If $\varphi$ is a state on $\mathcal{B}$ such that the restriction of $\varphi$ to $\mathcal{B}$ is a pure state, then $\varphi(q) = 0$.

**Proof.** Since $\varphi$ is a pure state on the commutative C*-algebra $\mathcal{B}$, $\varphi$ is multiplicative on $\mathcal{B}$. Let $\varphi(a) = \lambda$. Then by the Cauchy-Schwartz inequality

$$|\varphi(as) - \varphi(a)\varphi(s)|^2 = |\varphi((a - \lambda e)s)|^2 \leq \varphi((a - \lambda e)^2)\varphi(s*s) = 0,$$

and similarly, $\varphi(sa) = \varphi(a)\varphi(s)$. Hence, $\varphi(q) = \varphi(as - sa) = 0$.

If $s$ is the unilateral shift on $l^2$ ($s(\lambda_1, \lambda_2, \ldots) = (0, \lambda_1, \lambda_2, \ldots)$), then $s*s - ss^* = q$ is the projection of $l^2$ onto the space of sequences of the form $(\lambda_1, 0, 0, \ldots)$. If $a = s + s^*$, then $q = as - sa$, so $q$, $a$ and $s$ satisfy the hypotheses of the proposition. Hence, we may take $\mathcal{A}$ to be the C*-algebra generated by $s$ and $\mathcal{B}$ to be the C*-algebra generated by $a$ and $e$.

**References**


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