CHARACTERIZATIONS OF THE SPHERE
BY THE CURVATURE
OF THE SECOND FUNDAMENTAL FORM
UDO SIMON

Abstract. On an ovaloid S with Gaussian curvature \( K(I) > 0 \) in Euclidean three-space \( E^3 \) the second fundamental form defines a nondegenerate Riemannian metric with curvature \( K(II) \). R. Schneider [7] proved that the spheres in Euclidean space \( E^{n+1} \) are the only closed hypersurfaces on which the second fundamental form defines a nondegenerate Riemannian metric of constant curvature. For surfaces in \( E^3 \) we give a common generalization of Schneider’s theorem and the classical theorem of Liebmann [6] (which states that any ovaloid in \( E^3 \) with constant Gaussian curvature is a sphere).

We introduce the following notations. Let \( x: S^n \to E^{n+1} \) be an imbedding so that \( S = x(S^n) \) is an ovaloid in Euclidean \((n + 1)\)-space.

By appropriate orientation the second fundamental form \( II \) defines a Riemannian metric. Let \((u^i)\) be local coordinates and let \( \Gamma(I)_{ij}^k, K(I), \nabla (I), \nabla_I \) resp. \( \Gamma(II)_{ij}^k, K(II), \nabla (II), \nabla_{II} \) denote Christoffel symbols, Gaussian curvature, covariant differentiation and the first Beltrami operator with respect to the first fundamental form \( I \) respective to the second fundamental form \( II \). Let \( H \) be the mean curvature and

\[ T^k_{ij} = \Gamma(I)_{ij}^k - \Gamma(II)_{ij}^k. \]

In the following we shall use the second fundamental tensor \( b_{ij} \) for “raising and lowering the indices”. Then ([7, p. 232]) \( T_{ijk} := T^k_{ij} b_{hk} \) is totally symmetric and for \( n = 2 \) we have

\[ K(II) = H + \frac{1}{2} T_{ijk} T_{ijk} - \left( \frac{1}{8} K^2 \right) \cdot \nabla_{II} K(I), \]

which easily implies ([2, p. 7])

\[ 2H \left( K(II) - H \right) (H^2 - K(I)) \]

\[ = \frac{1}{2} K(I) \nabla_{II} \left( H, H^2 / K(I) \right) - \frac{1}{4} \nabla_I \left( H^2 / K(I), K(I) \right). \]

The following result is a simple consequence of the Gauss-Bonnet-integralformula in case \( n = 2 \).

3. Lemma ([5, Koutroufiotis]). Let \( S \) be an ovaloid in \( E^3 \). Then each of the assumptions

\((3a) \ K(II) > \left( K(I) \right)^{1/2}, \)

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(3b) \(K(II) < (K(I))^{1/2}\) implies \(K(II) = (K(I))^{1/2}\) on \(S\).

For the following result of Koutroufiotis we will give another proof.

4. Lemma. Let \(S\) be an ovaloid \((K(I) > 0)\) in \(E^3\). Then \(K(II) = (K(I))^{1/2}\) on \(S\) implies that \(S\) is a sphere.

Proof. Define \(g: S \rightarrow \mathbb{R}\) by \(g(q) = H^2(q)/K(I)(q), \quad q \in S\). We have \(g(q) \geq 1\) for \(q \in S\) and \(g(q_0) = 1\) if and only if \(q_0\) is an umbilic. Assume \(S\) not to be a sphere. Then there exists \(\bar{q} \in S\) with \(1 < g(\bar{q}) = \max_{q \in S} g(q)\) and \((\partial g/\partial u')(\bar{q}) = 0\), so that the right-hand side of (2) vanishes in \(\bar{q}\). On the other hand \(K(II) = (K(I))^{1/2}\) implies that the left-hand side of (2) is negative in \(\bar{q}\), as \((H^2 - K(I)(\bar{q})) > 0\), which is a contradiction. So \(g \equiv 1\) on \(S\) and \(S\) is a sphere.

5. Corollary. Let \(S\) be an ovaloid.

(5.1) Each of the assumptions (3a), (3b) implies that \(S\) is a sphere [5].

(5.2) \(K(II) \geq H\) implies that \(S\) is a sphere (cf. [3, Problem (9.6.b), p. 224] for \(n = 2\)).

The following lemma is an analogue to a classical theorem of Hilbert [4, Anhang V].

6. Lemma. Let \(S\) be an ovaloid in \(E^3\). If there exists \(q_0 \in S\) where \(A''(II)\) takes its minimum and \(A'(I)\) takes its maximum, then \(S\) is a sphere.

Proof. As \(T_{\alpha k} T^{\alpha k} > 0\) and \(H > (K(I))^{1/2}\), (1) implies

\[K(II) > (K(I))^{1/2} - \frac{1}{8K^2} \nabla_{II}(K(I)).\]

As \(\nabla_{II}(K(I))(q_0) = 0\) we get, for every \(q \in S\),

\[K(II)(q) > K(II)(q_0) > (K(I)(q_0))^{1/2} > (K(I)(q))^{1/2},\]

the assertion follows from (5.1).

Theorem. Let \(S\) be an ovaloid in \(E^3\). If there exists a function \(\Phi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) which is increasing (resp. decreasing) in both variables and strictly monotonic in at least one of its variables and if

\[\Phi(K(I)(q), K(II)(q)) = 0\]

for all \(q \in S\), then \(S\) is a sphere.

Proof. Assume \(\Phi\) to be strictly increasing in the second variable and assume that there exists \(q_0 \in S\) such that \(K(I)(q_0) = \max_{q \in S} K(I)(q)\)

but \(K(II)(q_0) > \min_{q \in S} K(II)(q) = K(II)(q_1)\). Then

\[0 = \Phi(K(I)(q_0), K(II)(q_0)) > \Phi(K(I)(q_0), K(II)(q_1)) > \Phi(K(I)(q_1), K(II)(q_1)) = 0,\]

which is a contradiction. So \(K(II)(q_0) = \min_{q \in S} K(II)(q)\) and the assertion follows from Lemma 6.
References


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