HYPERFINITENESS AND THE HALMOS-ROHLIN THEOREM FOR NONSINGULAR ABELIAN ACTIONS

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Abstract. Theorem 1. Let the countable abelian group $G$ act nonsingularly and aperiodically on Lebesgue space $(X, \mu)$. Then for each finite subset $A \subset G$ and $\epsilon > 0$ there exist finite $B \subset G$ and $F \subset X$ with $(bF : b \in B)$ disjoint and $\mu[(\bigcap_{a \in A} B - a)F] > 1 - \epsilon$.

Theorem 2. Every nonsingular action of a countable abelian group on a Lebesgue space is hyperfinite.

1. Introduction. The principal results here are a Halmos-Rohlin theorem for nonsingular actions of a countable abelian group on a Lebesgue measure space, and a proof of their hyperfiniteness. The latter fact has relevance for the group-measure space construction of von Neumann algebras. This construction produces algebras of type III precisely when there is no equivalent measure preserved by the action (see [9, Chapter 4.2]).

These results have already been proved for measure-preserving actions. Katznelson and Weiss [4] and Conze [1] proved a Halmos-Rohlin theorem for measure-preserving actions of $\mathbb{Z}^d$, and Krieger [5] extended this to countable abelian groups. Hyperfiniteness was shown in the measure-preserving case by Dye in the second of his pioneering papers [2] and [3]. However, it seems worthwhile to give a simpler proof of hyperfiniteness even in this case. Finally, Versik [10] has announced a proof of the hyperfiniteness of nonsingular countable abelian actions. However, the only proof of his of which we are aware [11] has serious gaps.

After this paper was completed, we learned from A. Connes that he and W. Krieger have also proved Theorem 2, apparently by somewhat different methods.

2. The Halmos-Rohlin theorem. All transformations act on a fixed Lebesgue measure space $(X, \mu)$ (see [8] for the properties of such spaces). An invertible measurable transformation of $X$ is called nonsingular if both it and its inverse map $\mu$-null sets to $\mu$-null sets. The group of all such transformations is denoted by $\mathcal{U}(\mu)$.

Let $G$ be a countable abelian group. A nonsingular action of $G$ on $(X, \mu)$ is a homomorphism $T : G \to \mathcal{U}(\mu)$. We abbreviate $T(g)(x)$ by $gx$, $T(g)(F)$ by...
gF for subsets F of X, etc. We say G acts aperiodically on (X, μ) if the only element of G which has a fixed point is the identity of G. If B ⊆ G and F ⊆ X, then BF denotes \( \cap \{ bF : b \in B \} \). A subset F of X is called a B set if \( \{ bF : b \in B \} \) is a disjoint collection. If A and B are subsets of G, then \( \cap_A B \) denotes \( \cap \{ B - a : a \in A \} \). We are now prepared to state the result of this section.

**Theorem 1.** Let the countable abelian group G act nonsingularly and aperiodically on the Lebesgue space (X, μ). Then for each finite subset A of G and each \( \varepsilon > 0 \), there exists a finite subset B of G and a measurable B set F with 

\[ \mu(\cap_A B)F > 1 - \varepsilon. \]

**Proof.** The proof builds on some ideas in [7]. For measure-preserving actions this proof is actually quite simple. Most of our proof is concerned with using averaging arguments to control the size of sets which are easily shown to be small for measure-preserving actions.

First, suppose that the theorem is true for actions of the d-dimensional integers \( \mathbb{Z}^d \). If G is a countable abelian group acting on X, and A is a finite subset of G, we can assume without loss that A generates G. Hence for some integer d and finite group H, we have G isomorphic to \( \mathbb{Z}^d \oplus H \). Let \( \xi \) be the partition of X into orbits of H, and \( q : X \to X/\xi \) be the quotient map. Then \( \xi \) is a measurable partition in the sense of Rohlin [8], and \( X/\xi \) is a Lebesgue space under the measure \( \mu(E) = \mu(q^{-1}E) \). The aperiodic action of \( \mathbb{Z}^d \) on X induces one on \( X/\xi \). Let \( A_0 \) be the projection of A into \( \mathbb{Z}^d \). By our initial assumption, for every \( \varepsilon > 0 \) there is a \( B_0 \subset \mathbb{Z}^d \) and \( F_0 \subset X/\xi \) with

\[ \mu_\xi(\cap_{A_0} B_0)F_0 > 1 - \varepsilon. \]

Let \( F \subset X \) be a measurable cross section of \( q \) restricted to \( q^{-1}(F_0) \), and \( B = B_0 \oplus H \). Then \( F \) is a B set, and since

\[ \cap_A B = (\cap_{A_0} B_0) \oplus H, \]

we have

\[ \mu\left( \left( \cap_A B \right)F \right) = \mu_\xi\left( \left( \cap_{A_0} B_0 \right)F_0 \right) > 1 - \varepsilon. \]

Thus we may assume that \( G = \mathbb{Z}^d \). The basic strategy is the following. We begin by showing that for arbitrarily large cubes \( Q \) in \( \mathbb{Z}^d \) there is a Q set F such that for every \( t \) in a cube one fourth the size of \( Q \), the set \( (Q + t)F \) fills out at least a fixed proportion of \( F \). The same argument is applied to fill out a fixed proportion of the remainder of X with a smaller cube, and these two constructions are combined to fill out a larger proportion of the entire space using the smaller cube. This combination involves an averaging argument which uses some flexibility in the original choice of \( Q \). Repeating this procedure eventually fills out as much of X as desired.

Let \( Q_p \) be the cube \( (0, 1, \ldots, P - 1)^d \), and

\[ R_p = \{-P, -P + 1, \ldots, P - 1\}^d, \]

so that \( R_p \) is made up of \( 2^d \) translates of \( Q_p \). Let \( B_N(Q_p) \) be a barrier of thickness \( N \) surrounding \( Q_p \), namely

\[ B_N(Q_p) = \{ t \in \mathbb{Z}^d : -N \leq t_j < P + N \} \setminus Q_p. \]

If \( I \) divides \( P \), let \( S_I(Q_p) \) be \( Q_p \) “shrunk” symmetrically so that a proportion
1/L is removed from its surface, namely \( S_L(Q_p) = \{ t \in \mathbb{Z}^d : P/L < t_j < P - P/L \} \).

Say an integer \( L \) works for the real number \( B \) if there is some \( M \) such that for all multiples \( P \) of \( LM \) there exists a \( Q_p \) set \( F \) with \( \mu(S_L(Q_p)F) > \beta \). Let \( \alpha \) be the supremum of the numbers \( \beta \) for which some integer works. We will show that \( \alpha = 1 \). This will prove the theorem, since for a finite subset \( A \) of \( G \), we have \( S_L(Q_p) \subset \cap A Q_p \) for sufficiently large \( P \).

We first show that \( 4 \) works for \( 4^{-d} \), so that \( \alpha > 0 \). Aperiodicity of the action guarantees that for every integer \( P \) and every subset \( E \) of positive measure, there is a \( Q_p \) set of positive measure contained in \( E \). Zorn's Lemma provides a maximal \( Q_p \) set \( F \); that is, a measurable subset \( F \) such that if \( F' \supseteq F \) and \( F' \) is also a \( Q_p \) set, then \( \mu(F' \setminus F) = 0 \). We claim that \( \mu(R_p F) = 1 \). For otherwise, \( X \setminus R_p F \) would contain a \( Q_p \) set of positive measure, and this could clearly be combined with \( F \) to produce a larger \( Q_p \) set. Now assume that \( P \) is divisible by 4. Then \( R_p \) is covered by \( 4^d \) translates of \( Q_{P/2} \), and so there must be at least one such translate, say \( Q_{P/2} + t \), for which \( \mu((Q_{P/2} + t)F) > 4^{-d} \). Choose \( t' \) so that \( Q_{P/2} + t - t' = S_d(Q_p) \). Set \( F_1 = t'F \). Then \( F_1 \) is a \( Q_p \) set and \( \mu(S_d(Q_p)F_1) = \mu((Q_{P/2} + t)F) > 4^{-d} \).

Now suppose that \( \beta < \alpha \). Let \( \xi, \eta \) be positive numbers. We will show that \( \beta - \xi + 4^{-d}(1 - \beta - \eta) \) is also less than \( \alpha \). Hence \( \beta + 4^{-d}(1 - \beta) < \alpha \). But the only number \( \alpha \) in \( (0, 1] \) which can have this property for all \( \beta < \alpha \) is \( \alpha = 1 \).

The proof of our assertion is based on the following two statements.

(i) If \( \eta > 0 \), then for every sufficiently large even integer \( M \), integer \( L \), multiple \( P = NLM \) of \( LM \), and \( Q_p \) set \( F \), for over \( 9/10 \) of the elements \( t \) in \( Q_{P/L} \) we have

\[
\mu\left[ B_{2N}(Q_p)(tF) \right] < \eta.
\]

(ii) If \( \xi > 0 \), then for every sufficiently large integer \( L \), integer \( M \), multiple \( P = NLM \) of \( LM \) by a multiple \( N \) of \( L \), and \( Q_p \) set \( F \), for over \( 9/10 \) of the \( t \) in \( Q_{P/L} \) we have

\[
\mu\left[ S_L(Q_N)(NQ_{P/N})(tF) \right] > \mu\left[ Q_p(tF) \right] - \xi.
\]

We will prove statement (i). The proof of (ii) is similar, and we omit it. Divide \( B_{2N}(Q_p) \) into lower dimensional "slabs" as follows. For each nonempty subset \( \Lambda \) of \( \{1, 2, \ldots, d\} \) and each \( \sigma: \Lambda \to \{ -1, 1 \} \), let

\[
B_{\Lambda}^\sigma = \{ t: 0 < t_j < P \text{ if } j \not\in \Lambda, -2N < t_j < 0 \text{ if } \sigma(j) = -1, P < t_j < P + 2N \text{ if } \sigma(j) = 1 \}.
\]

Then \( B_{2N}(Q_p) \) is the disjoint union of the \( B_{\Lambda}^\sigma \). The number of these slabs is easily seen to be \( 3^d - 1 \). Let

\[
Q_{\Lambda} = \{ 2Nt: t_j = 0 \text{ if } j \not\in \Lambda, 0 < t_j < M/2 \text{ if } j \in \Lambda \}.
\]

Since \( B_{\Lambda}^\sigma + Q_{\Lambda} \) is contained in a translate of \( Q_p \), and since the collection \( \{ B_{\Lambda}^\sigma + t: t \in Q_{\Lambda} \} \) is disjoint, it follows that \( B_{\Lambda}^\sigma F \) is a \( Q_{\Lambda} \) set.

Let the cardinality of a set \( A \) be denoted by \( |A| \). Then \( Q_{P/L} \) consists of a disjoint union of \( |Q_{P/L}|/|Q_{\Lambda}| \) translates of \( Q_{\Lambda} \).
\[
\sum_{t \in Q_{P/L}} \mu\left[ B_{2N}^{\sigma} (tF) \right] \leq \frac{|Q_{P/L}|}{|Q_{\lambda}|}.
\]

Since \(|Q_{\lambda}| > M/2\), the right-hand side is bounded by \(2|Q_{P/L}|/M\). Summing over \(\Lambda\) and \(\sigma\) shows that
\[
\sum_{t \in Q_{P/L}} \mu\left[ B_{2N} (Q_{P}) (tF) \right] \leq \frac{2(3^d - 1)|Q_{P/L}|}{M},
\]
and hence that
\[
\frac{1}{|Q_{P/L}|} \left\{ t \in Q_{P/L} : \mu\left[ B_{2N} (Q_{P}) (tF) \right] > \frac{20(3^d - 1)}{M} \right\} < \frac{1}{10}.
\]

If \(M > 20(3^d - 1)/\eta\), the desired inequality in (i) holds.

We now complete the proof of the theorem using (i) and (ii). Suppose \(\beta < \alpha\), and choose \(L_1\) to work for \(\beta\). This means that there exists an \(M_1\) such that for any multiple \(P\) of \(L_1M_1\) there is a \(Q_{P}\) set \(F\) with \(\mu[S_L(Q_{P}) F] > \beta\). Let \(M\) be an even multiple of \(M_1\) and so large that (i) holds. Let \(L\) be a multiple of \(L_1\) and so large that (ii) holds. Let \(P\) be a multiple of \(LM\) by a multiple \(N\) of \(2L\). Hence for the \(Q_{P}\) set \(F\) with \(\mu[S_L(Q_{P}) F] > \beta\), there exists some \(t\) in \(Q_{P/L}\) so that the inequalities in both (i) and (ii) hold. Let \(F_1 = tF\), and let
\[E = X \setminus (Q_{P} \cup B_{2N} (Q_{P})) F_1.\]

Choose a maximal \(Q_{N}\) set \(F_2\) in \(E\). Then, arguing as before, \(E \subset R_N F_2\) except for a null set. Now \(R_N\) is the union of \(4^d\) translates of \(Q_{N/2}\), so for one of these, say \(Q_{N/2} + u\), we must have
\[
\mu\left[ \left\{ (Q_{N/2} + u) F_2 \right\} \cap E \right] > 4^{-d} \mu(E).
\]

Choose \(u'\) so that \(Q_{N/2} + u = S_4(Q_{N})\), and put \(F_3 = u' F_2\). Finally, put \(F_4 = (N Q_{P/N}) F_1\). We will check that \(F' = F_3 \cup F_4\) is a \(Q_{N}\) set for which
\[
\mu\left[ S_L (Q_{N}) F' \right] > \beta - \xi + 4^{-d}(1 - \beta - \eta).
\]

This will show that \(L\) works for \(\beta - \xi + 4^{-d}(1 - \beta - \eta)\), and complete the proof.

The set \(F_2\) was chosen to be a \(Q_{N}\) set, so the same holds for \(F_3\). Since \(F_1\) is a \(Q_{P}\) set, it follows that \(F_4\) is a \(Q_{N}\) set, and
\[Q_N F_4 = (Q_N + N Q_{P/N}) F_1 = Q_P F_1.\]

Now \(u' \in R_N\), so that
\[Q_N F_3 = Q_N (u' F_2) \subset R_{2N} E.\]

Since \(R_{2N} E\) is disjoint from \(Q_P F_1\), we have that \(F_3 \cup F_4\) is also a \(Q_{N}\) set.

We estimate the measures of the disjoint sets \(S_L(Q_{N}) F_3\) and \(S_L(Q_{N}) F_4\) separately. By (i) we have
\[
\mu\left[ S_L (Q_{N}) F_3 \right] > \mu\left[ S_4 (Q_{N}) F_3 \right] = \mu\left[ (Q_{N/2} + u) F_2 \right] > 4^{-d} \mu(E)
\]

\[
> 4^{-d}(1 - \mu\left[ Q_P (tF) \right] - \mu\left[ B_{2N} (Q_{P}) (tF) \right])
\]

\[> 4^{-d}(1 - \mu\left[ Q_P (tF) \right] - \eta).\]
Also, using (ii) we have
\[ \mu\left[ S_L(Q_N)F_A \right] = \mu\left[ S_L(Q_N)(NQ_{P/N})(tF) \right] \geq \mu\left[ Q_P(tF) \right] - \xi. \]
Thus
\[ \mu\left[ S_L(Q_N)F' \right] \geq \mu\left[ Q_P(tF) \right] - \xi + 4^{-d}(1 - \mu\left[ Q_P(tF) \right] - \eta). \]
Since \( t \in Q_{P/L} \), we have \( Q_P + t \supset S_L(Q_P) \), so that
\[ \mu\left[ Q_P(tF) \right] \geq \mu\left[ S_L(Q_P)F \right] \geq \mu\left[ S_L(Q_P)F \right] \geq \beta. \]
Applying this to the right side of the previous inequality gives the desired result.

3. Hyperfiniteness. A nonsingular action of a countable group \( G \) on \( X \) is called hyperfinite if for each finite subset \( A \) of \( G \) and each \( \varepsilon > 0 \), there exists some finite group \( K \subset \mathcal{H}(\mu) \) such that \( Kx \subset Gx \) for almost every \( x \), and such that for each \( a \in A \) there is some \( k \in K \) with \( \mu(\{x: ax \neq kx\}) < \varepsilon \).

This definition (in the measure-preserving case) is due to Dye [2]. Two equivalent definitions are the following:

(1) there is some nonsingular action of \( \mathbb{Z} \) on \( X \) such that \( \mathbb{Z}x = Gx \) for almost every \( x \);

(2) there exist finite groups \( G_1 \subset G_2 \subset \cdots \) of nonsingular transformations of \( X \) with \( \bigcup G_nx = Gx \) for almost every \( x \).

The proof of the equivalence of these with the original definition is in [2] and [5].

The first lemma describes the aperiodic decomposition of \( X \).

**Lemma 1.** Let the countable abelian group \( G \) act nonsingularly on \( X \). If \( H \) is a subgroup of \( G \), let \( X_H = \{x: gx = x \text{ if and only if } g \in H\} \). Then \( X \) is the disjoint union of the \( X_H \), each \( X_H \) is measurable and invariant under \( G \), and \( G/H \) acts aperiodically on \( X_H \).

**Proof.** Clear.

**Lemma 2.** A nonsingular action of \( G \) is hyperfinite if for each finite subset \( A \) of \( G \) and each \( \varepsilon > 0 \), there exists a finite subset \( B \) of \( G \) and a \( B \) set \( F \) in \( X \) such that \( \mu(\bigcap A B)F \geq 1 - \varepsilon \).

**Proof.** Suppose that \( A \) is a finite subset of \( G \) which contains the identity, and let \( \varepsilon > 0 \). Choose \( B \) and \( F \) to satisfy the hypothesis. We construct the required finite group \( K \) from \( B \) and \( F \) as follows. For each permutation \( \pi \) of \( B \), let \( T_\pi \in \mathcal{H}(\mu) \) be defined by
\[ T_\pi x = \begin{cases} \pi(b)x & \text{if } x \in bF (b \in B), \\ x & \text{if } x \in X \setminus BF. \end{cases} \]

The collection of such \( T_\pi \) forms a finite group \( K \) in \( \mathcal{H}(\mu) \). Clearly \( Kx \subset Gx \) for every \( x \). If \( a \in A \), the map \( b \mapsto b + a \) from \( \bigcap A B \) to \( B \) extends to a permutation \( \sigma_a \) of \( B \). Then \( T_{\sigma_a} \in K \), and
\[ \mu(\{x: ax \neq T_{\sigma_a}x\}) \geq \mu(\bigcap A B)F \geq 1 - \varepsilon. \]

**Theorem 2.** Every nonsingular action of a countable abelian group on a Lebesgue space is hyperfinite.
Proof. From the definition of hyperfiniteness, it is clear that it suffices to consider the case where \( G \) is finitely generated. Since the number of subgroups of a finitely generated abelian group is countable, by Lemma 1 it suffices to consider aperiodic actions. The result then follows from Theorem 1 and Lemma 2.

Remark. Lemma 2 gives a criterion for hyperfiniteness of nonsingular actions of countable groups which are not necessarily abelian. Our results show that all countable abelian groups satisfy this criterion. In the measure-preserving case, the conditions on \( B \) and \( F \) can be replaced by \( \mu(BF) > 1 - \varepsilon \) and \( |\bigcap A B| > (1 - \varepsilon)|B| \), the latter being a condition only on the group and not the action.

References


