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THE LOCAL RESOLVENT SET OF A LOCALLY LIPSCHITZIAN TRANSFORMATION IS OPEN

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ABSTRACT. The purpose of this paper is to prove that if p is a point of a complex Banach space H at which the nonlinear transformation T on H is locally Lipschitzian, then the local resolvent set of T at p is open.

Denote by H a complex Banach space with nondegenerate point-set S and norm $\|\cdot\|$ and by p a point of S. Denote by I the identity transformation on H and by T a not necessarily linear transformation from a subset $D(T)$ of S into S. If $D(T) = S$ and T is bounded and linear, then the resolvent set $\rho(T)$ of T is open [1, pp. 86–89]. Since in this case the local resolvent set $\rho_p(T)$ of T at p [2, pp. 212, 213] is $\rho(T)$, this means that $\rho_p(T)$ is open. In [2, Theorem 2, pp. 213–215] the openness of $\rho_p(T)$ is extended to the case where T is not necessarily linear but continuously Fréchet differentiable on an open set containing p and H is finite-dimensional. The following theorem establishes the fact that neither the differentiability of T nor the finite dimensionality of H is necessary to the openness of $\rho_p(T)$.

THEOREM. *If T is locally Lipschitzian at p, then $\rho_p(T)$ is open.*

PROOF. Denote by a a member of $\rho_p(T)$ and by $(\delta_a, \varepsilon_a)$ a positive-number pair such that [2, pp. 212, 213]

(1) $aI - T$ is 1-1 on the ball $R_p(\delta_a)$ with center p and radius δ_a;

(2) the ball $R_{(aI-T)p}(\varepsilon_a) \subseteqq (aI - T)(R_p(\delta_a))$;

(3) $(aI - T|_{R_p(\delta_a)})^{-1}$ is Lipschitzian on $R_{(aI-T)p}(\varepsilon_a)$ ($T|_{R_p(\delta_a)}$ is the restriction of T to $R_p(\delta_a)$).

Properties (1)–(3) permit us to denote by (r, M) a positive-number pair such that $\|(aI - T)x - (aI - T)y\| \geqq M\|x - y\|$ whenever $\{x, y\} \subseteqq R_p(r)$. Let $A = aI - T|_{R_p(\delta_a)}$, and denote by $|A^{-1}|$ the least nonnegative number B such that $\|A^{-1}x - A^{-1}y\| \leqq B\|x - y\|$ whenever $\{x, y\} \subseteqq R_{Ap}(\varepsilon_a)$. Finally, let

$$c = \min\{1/(2|A^{-1}|), M, \varepsilon_a/3, \delta_a/[|A^{-1}|(2 + \delta_a)], r/[|A^{-1}|(2 + r)]\},$$

and suppose that b is a complex number such that $|a - b| < c$. The remainder of the proof will be devoted to showing that b is in $\rho_p(T)$.

Let $\delta_b = \min\{r, \delta_a\}$ and $\varepsilon_b = \min\{\varepsilon_a/2, c\}$. If each of x and y is in $R_p(\delta_b)$, then

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$$\begin{aligned}
&\|(bI - T)x - (bI - T)y\| \\
&\quad = \|bx - Tx + ax - ax + ay - ay - by + Ty\| \\
&\quad = \|(aI - T)x - (aI - T)y - (a - b)(x - y)\| \\
&\quad \geqq \|(aI - T)x - (aI - T)y\| - |a - b|\,\|x - y\| \\
&\quad \geqq (M - |a - b|)\,\|x - y\| \quad \text{since } \delta_b \leqq r.
\end{aligned}$$

Since $|a - b| < c \leqq M$, we have that $M - |a - b| > 0$. Thus $bI - T$ is 1-1 on $R_p(\delta_b)$ and $(bI - T|_{R_p(\delta_b)})^{-1}$ is Lipschitzian on $(bI - T)(R_p(\delta_b))$. It remains to be shown that $R_{(bI-T)p}(\varepsilon_b) \subseteqq (bI - T)(R_p(\delta_b))$.

Denote by y a point of $R_{(bI-T)p}(\varepsilon_b)$. To show that y is in $(bI - T)(R_p(\delta_b))$, we shall prove that the restriction of the transformation $A^{-1}((a - b)I + y)$ to $R_p(\delta_b)$ has a fixed point. The technique used is successive approximation.

Define a sequence u as follows. Let $u_0 = p$. Then

$$\begin{aligned}
\|(a - b)u_0 + y - Ap\| &= \|(a - b)p + y - (aI - T)p\| \\
&= \|y - (bI - T)p\| \\
&< \varepsilon_b < \varepsilon_a \quad \text{by choice of } \varepsilon_b.
\end{aligned}$$

Thus $(a - b)u_0 + y$ is in $R_{Ap}(\varepsilon_a)$. Since $R_{Ap}(\varepsilon_a) \subseteqq A(R_p(\delta_a))$ by (2), let $u_1 = A^{-1}((a - b)u_0 + y)$. Thus

$$\begin{aligned}
\|u_1 - u_0\| &= \|A^{-1}((a - b)u_0 + y) - A^{-1}Ap\| \\
&= \|A^{-1}((a - b)p + y) - A^{-1}Ap\| \\
&\leqq |A^{-1}|\,\|(a - b)p + y - Ap\| \\
&< |A^{-1}|\varepsilon_b \leqq |A^{-1}|c \quad \text{by choice of } \varepsilon_b.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|(a - b)u_1 + y - Ap\| &= \|(a - b)u_1 - (a - b)u_0 + (a - b)u_0 + y - Ap\| \\
&\leqq |a - b|\,\|u_1 - u_0\| + \|(a - b)u_0 + y - Ap\| \\
&< |a - b|\,|A^{-1}|c + \varepsilon_b < |A^{-1}|c^2 + c.
\end{aligned}$$

By definition of c we have that $|A^{-1}|c \leqq 1/2$, so $|A^{-1}|c^2 + c \leqq (3/2)c \leqq (3/2)(\varepsilon_a/3) < \varepsilon_a$. Thus $(a - b)u_1 + y$ is in $R_{Ap}(\varepsilon_a)$, so denote by u_2 the point $A^{-1}((a - b)u_1 + y)$. Therefore

$$\begin{aligned}
\|u_2 - u_1\| &= \|A^{-1}((a - b)u_1 + y) - A^{-1}((a - b)u_0 + y)\| \\
&\leqq |A^{-1}|\,|a - b|\,\|u_1 - u_0\| \\
&< (|A^{-1}|c)^2 \quad \text{since } |a - b| < c \text{ and } \|u_1 - u_0\| < |A^{-1}|c.
\end{aligned}$$

This implies the following:

$$\|u_2 - p\| \leqq \|u_2 - u_1\| + \|u_1 - p\| < (|A^{-1}|c)^2 + |A^{-1}|c.$$

Finally, we have

$$\begin{aligned}\|(a-b)u_2+y-Ap\| &= \|(a-b)u_2-(a-b)u_1+(a-b)u_1+y-Ap\| \\ &\leqq |a-b|\,\|u_2-u_1\|+\|(a-b)u_1+y-Ap\| \\ &< c(|A^{-1}|c)^2+c\sum_{i=0}^{1}(|A^{-1}|c)^i \\ &= c\sum_{i=0}^{2}(|A^{-1}|c)^i.\end{aligned}$$

For the inductive step, suppose that m is an integer not less than 2 and that $u_0, u_1, \ldots, u_m$ has the following properties:

(4) if k is an integer in $[1,m]$, then

$$\|u_k-u_{k-1}\| < (|A^{-1}|c)^k \quad \text{and} \quad \|u_k-p\| < \sum_{i=1}^{k}(|A^{-1}|c)^i;$$

(5) if k is an integer in $[0,m]$, then

$$\|(a-b)u_k+y-Ap\| < c\sum_{i=0}^{k}(|A^{-1}|c)^i;$$

(6) $$u_0 = p \quad \text{and} \quad u_k = A^{-1}((a-b)u_{k-1}+y)$$

for each integer k in $[1,m]$.

Since $|A^{-1}|c \leqq 1/2$, we have that $c\sum_{i=0}^{m}(|A^{-1}|c)^i \leqq 2c \leqq 2(\varepsilon_a/3) < \varepsilon_a$. Thus $(a-b)u_m+y$ is in $R_{Ap}(\varepsilon_a)$, which is a subset of the domain of A^{-1} by (2); so let $u_{m+1} = A^{-1}((a-b)u_m+y)$. Then

$$\begin{aligned}\|u_{m+1}-u_m\| &= \|A^{-1}((a-b)u_m+y)-A^{-1}((a-b)u_{m-1}+y)\| \\ &\leqq |A^{-1}|\,|a-b|\,\|u_m-u_{m-1}\| \\ &< |A^{-1}|c(|A^{-1}|c)^m \quad \text{by (4)} \\ &= (|A^{-1}|c)^{m+1}.\end{aligned}$$

In addition,

$$\begin{aligned}\|u_{m+1}-p\| &\leqq \|u_{m+1}-u_m\|+\|u_m-p\| \\ &< (|A^{-1}|c)^{m+1}+\sum_{i=1}^{m}(|A^{-1}|c)^i \\ &\qquad \text{by the preceding inequality and (4)} \\ &= \sum_{i=1}^{m+1}(|A^{-1}|c)^i.\end{aligned}$$

Finally,

$$\begin{aligned}\|(a-b)u_{m+1}+y-Ap\| &\leqq |a-b|\,\|u_{m+1}-u_m\|+\|(a-b)u_m+y-Ap\| \\ &< c(|A^{-1}|c)^{m+1}+c\sum_{i=0}^{m}(|A^{-1}|c)^i \\ &= c\sum_{i=0}^{m+1}(|A^{-1}|c)^i.\end{aligned}$$

This completes the inductive step and defines a sequence u with the following properties:

(7) if n is a positive integer, then

$$\|u_n - u_{n-1}\| < (|A^{-1}|c)^n \quad \text{and} \quad \|u_n - p\| < \sum_{i=1}^{n} (|A^{-1}|c)^i;$$

(8) if n is a nonnegative integer, then

$$\|(a-b)u_n + y - Ap\| < c \sum_{i=0}^{n} (|A^{-1}|c)^i;$$

$$u_0 = p \quad \text{and} \quad u_n = A^{-1}((a-b)u_{n-1} + y) \tag{9}$$

for each positive integer n.

Since $|A^{-1}|c \leqq 1/2$, we know that $\{\sum_{i=0}^{n} (|A^{-1}|c)^i\}_{n=0}^{\infty}$ is convergent and, hence, u is Cauchy. Since H is complete, denote by x the sequential limit of u.

To show that x is in $R_p(\delta_b)$ and $(bI - T)x = y$, let us first prove that $(a-b)x + y$ is in $R_{Ap}(\varepsilon_a)$. If $d > 0$ and t is a positive integer such that $\|x - u_t\| < d/c$, we see that

$$\begin{aligned}
\|(a-b)x + y - Ap\| &\leqq |a-b|\,\|x - u_t\| + \|(a-b)u_t + y - Ap\| \\
&< c(d/c) + c \sum_{i=0}^{t} (|A^{-1}|c)^i \quad \text{by (8)} \\
&< d + 2c \\
&\leqq d + 2(\varepsilon_a/3) \quad \text{by the choice of } c.
\end{aligned}$$

Thus $\|(a-b)x + y - Ap\| \leqq 2(\varepsilon_a/3) < \varepsilon_a$. This means that, since A^{-1} is continuous by (3), we have

$$\begin{aligned}
x &= \lim_{n\to\infty} u_n = \lim_{n\to\infty} A^{-1}((a-b)u_{n-1} + y) \\
&= A^{-1}\Big[\lim_{n\to\infty} ((a-b)u_{n-1} + y)\Big] = A^{-1}\Big[(a-b)\lim_{n\to\infty} u_{n-1} + y\Big] \\
&= A^{-1}((a-b)x + y).
\end{aligned}$$

Therefore $Ax = (a-b)x + y$, so $ax - Tx = Ax = (a-b)x + y$ and $(bI - T)x = y$.

To complete the proof it must be shown that x is in $R_p(\delta_b)$. Since x is the sequential limit of u, we know by (7) that $\|x - p\| \leqq \sum_{i=1}^{\infty} (|A^{-1}|c)^i$, which is $|A^{-1}|c/(1 - |A^{-1}|c)$ by the fact that $|A^{-1}|c < 1$. The fact that

$$c \leqq \delta_a/[|A^{-1}|\,(2 + \delta_a)]$$

implies that $|A^{-1}|c/(1 - |A^{-1}|c) \leqq \delta_a/2$. Similarly, the choice of c to be not greater than $r/[|A^{-1}|(2 + r)]$ means that

$$\|x - p\| \leqq |A^{-1}|c/(1 - |A^{-1}|c) \leqq r/2.$$

Thus

$$\|x - p\| \leqq \min\{\delta_a/2, r/2\} < \min\{\delta_a, r\} = \delta_b,$$

so x is in $R_p(\delta_b)$ and y is in $(bI - T)(R_p(\delta_b))$. This completes the proof.

An example of the phenomenon described by the Theorem is the transformation I^* on the complex numbers, which maps each complex number onto its conjugate. I^* is Lipschitzian, hence locally Lipschitzian at each complex number. The local resolvent set of I^* at 0–indeed at any complex number–is the complement in the complex numbers of the unit circle.

The Theorem reveals a further similarity between the global spectrum of a bounded linear transformation and the local spectrum [2, pp. 212, 213] of certain nonlinear transformations. In doing so it heightens one's hope that there is a suitable local analog for nonlinear transformations to the spectral representation theory for bounded linear ones. One of the next questions to be answered in attempting to discover such a theory seems to be the following: Does a locally Lipschitzian transformation (or a continuously differentiable transformation on a non-finite-dimensional space) have a local spectrum? It is my feeling that the answer is affirmative.

References

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