THE LOCAL RESOLVENT SET OF A LOCALLY LIPSCHITZIAN TRANSFORMATION IS OPEN

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Abstract. The purpose of this paper is to prove that if \( p \) is a point of a complex Banach space \( H \) at which the nonlinear transformation \( T \) on \( H \) is locally Lipschitzian, then the local resolvent set of \( T \) at \( p \) is open.

Denote by \( H \) a complex Banach space with nondegenerate point-set \( S \) and norm \( \| \cdot \| \) and by \( p \) a point of \( S \). Denote by \( I \) the identity transformation on \( H \) and by \( T \) a not necessarily linear transformation from a subset \( D(T) \) of \( S \) into \( S \). If \( D(T) = S \) and \( T \) is bounded and linear, then the resolvent set \( \rho(T) \) of \( T \) is open [1, pp. 86–89]. Since in this case the local resolvent set \( \rho_p(T) \) of \( T \) at \( p \) [2, pp. 212, 213] is \( \rho(T) \), this means that \( \rho_p(T) \) is open. In [2, Theorem 2, pp. 213–215] the openness of \( \rho_p(T) \) is extended to the case where \( T \) is not necessarily linear but continuously Fréchet differentiable on an open set containing \( p \) and \( H \) is finite-dimensional. The following theorem establishes the fact that neither the differentiability of \( T \) nor the finite dimensionality of \( H \) is necessary to the openness of \( \rho_p(T) \).

**Theorem.** If \( T \) is locally Lipschitzian at \( p \), then \( \rho_p(T) \) is open.

**Proof.** Denote by \( a \) a member of \( \rho_p(T) \) and by \( (\delta_a, \epsilon_a) \) a positive-number pair such that [2, pp. 212, 213]

1. \( aI - T \) is 1-1 on the ball \( R_p(\delta_a) \) with center \( p \) and radius \( \delta_a \);
2. the ball \( R_{(aI - T)p}(\epsilon_a) \subseteq (aI - T)(R_p(\delta_a)) \);
3. \( (aI - T)\big|_{R_p(\delta_a)}^{-1} \) is Lipschitzian on \( R_{(aI - T)p}(\epsilon_a) \). (\( T\big|_{R_p(\delta_a)} \) is the restriction of \( T \) to \( R_p(\delta_a) \)).

Properties (1)–(3) permit us to denote by \( (r, M) \) a positive-number pair such that \( \| (aI - T)x - (aI - T)y \| \geq M \| x - y \| \) whenever \( (x, y) \subseteq R_p(r) \). Let \( A = aI - T\big|_{R_p(\delta_a)} \), and denote by \( |A^{-1}| \) the least nonnegative number \( B \) such that \( \| A^{-1}x - A^{-1}y \| \leq B \| x - y \| \) whenever \( (x, y) \subseteq R_{Ap}(\epsilon_a) \). Finally, let

\[
c = \min\left\{1/(2|A^{-1}|), M, \epsilon_a/3, \delta_a/(|A^{-1}|(2 + \delta_a)), r/(|A^{-1}|(2 + r))\right\},
\]

and suppose that \( b \) is a complex number such that \( |a - b| < c \). The remainder of the proof will be devoted to showing that \( b \) is in \( \rho_p(T) \).

Let \( \delta_b = \min\{r, \delta_a\} \) and \( \epsilon_b = \min\{\epsilon_a/2, c\} \). If each of \( x \) and \( y \) is in \( R_p(\delta_b) \), then...
\[ \|(bI - T)x - (bI - T)y\| \]
\[ = \|bx - Tx + ax - ax + ay - ay - by + Ty\| \]
\[ = \|((aI - T)x - (aI - T)y - (a - b)(x - y))\| \]
\[ \geq \|((aI - T)x - (aI - T)y\| - |a - b||x - y\| \]
\[ \geq (M - |a - b|)||x - y|| \quad \text{since } \delta_b \leq r. \]

Since \(|a - b| < c \leq M\), we have that \(M - |a - b| > 0\). Thus \(bI - T\) is 1-1 on \(R_p(\delta_b)\) and \((bI - T|R_p(\delta_b))^{-1}\) is Lipschitzian on \((bI - T)(R_p(\delta_b))\). It remains to be shown that \(R_{(bI - T)p}(\delta_b) \subseteq (bI - T)(R_p(\delta_b))\).

Denote by \(y\) a point of \(R_{(bI - T)p}(\delta_b)\). To show that \(y\) is in \((bI - T)(R_p(\delta_b))\), we shall prove that the restriction of the transformation \(A^{-1}((a - b)I + y)\) to \(R_{(bI - T)p}(\delta_b)\) has a fixed point. The technique used is successive approximation.

Define a sequence \(u\) as follows. Let \(u_0 = p\). Then
\[ \|(a - b)u_0 + y - Ap\| = \|(a - b)p + y - (aI - T)p\| \]
\[ = \|y - (bI - T)p\| \]
\[ < \epsilon_b < \epsilon_a \quad \text{by choice of } \epsilon_b. \]

Thus \((a - b)u_0 + y\) is in \(R_{Ap}(\epsilon_a)\). Since \(R_{Ap}(\epsilon_a) \subseteq A(R_p(\delta_a))\) by (2), let \(u_1 = A^{-1}((a - b)u_0 + y)\). Thus
\[ \|u_1 - u_0\| = \|A^{-1}((a - b)u_0 + y) - A^{-1}Ap\| \]
\[ = \|A^{-1}((a - b)p + y) - A^{-1}Ap\| \]
\[ \leq |A^{-1}||a - b||p + y - Ap\| \]
\[ < |A^{-1}||a - b|| + \epsilon_b < |A^{-1}|c \quad \text{by choice of } \epsilon_b. \]

Furthermore,
\[ \|(a - b)u_1 + y - Ap\| = \|(a - b)u_1 - (a - b)u_0 + (a - b)u_0 + y - Ap\| \]
\[ \leq |a - b||u_1 - u_0|| + ||(a - b)u_0 + y - Ap|| \]
\[ < |a - b||u_1 - u_0|| + \epsilon_b < |A^{-1}|c^2 + c. \]

By definition of \(c\) we have that \(|A^{-1}|c \leq 1/2\), so \(|A^{-1}|c^2 + c \leq (3/2)c \leq (3/2)(\epsilon_a/3) < \epsilon_a\). Thus \((a - b)u_1 + y\) is in \(R_{Ap}(\epsilon_a)\), so denote by \(u_2\) the point \(A^{-1}((a - b)u_1 + y)\). Therefore
\[ \|u_2 - u_1\| = \|A^{-1}((a - b)u_1 + y) - A^{-1}((a - b)u_0 + y)\| \]
\[ \leq |A^{-1}||a - b||u_1 - u_0|| \]
\[ < (|A^{-1}|c)^2 \quad \text{since } |a - b| < c \text{ and } \|u_1 - u_0\| < |A^{-1}|c. \]

This implies the following:
\[ \|u_2 - p\| \leq \|u_2 - u_1\| + \|u_1 - p\| < (|A^{-1}|c)^2 + |A^{-1}|c. \]
Finally, we have

\[\|(a - b)u_2 + y - Ap\| = \|(a - b)u_2 - (a - b)u_1 + (a - b)u_1 + y - Ap\|\]

\[\leq |a - b| \|u_2 - u_1\| + \|(a - b)u_1 + y - Ap\|\]

\[< c(|A^{-1}|c)^2 + c \sum_{i=0}^1 (|A^{-1}|c)^i\]

\[= c \sum_{i=0}^2 (|A^{-1}|c)^i.\]

For the inductive step, suppose that \(m\) is an integer not less than 2 and that \(u_0, u_1, \ldots, u_m\) has the following properties:

(4) if \(k\) is an integer in \([1,m]\), then

\[\|u_k - u_{k-1}\| < (|A^{-1}|c)^k\]

\[\text{and}\]

\[\|u_k - p\| < \sum_{i=1}^k (|A^{-1}|c)^i;\]

(5) if \(k\) is an integer in \([0,m]\), then

\[\|(a - b)u_k + y - Ap\| < c \sum_{i=0}^k (|A^{-1}|c)^i;\]

(6) \(u_0 = p\) and

\[u_k = A^{-1}((a - b)u_{k-1} + y)\]

for each integer \(k\) in \([1,m]\).

Since \(|A^{-1}|c \leq 1/2\), we have that \(c \sum_{i=0}^m (|A^{-1}|c)^i \leq |c| \leq 2(e_a/3) < e_a\).

Thus \((a - b)u_m + y\) is in \(R_{Ap}(e_a)\), which is a subset of the domain of \(A^{-1}\) by (2); so let \(u_m+1 = A^{-1}((a - b)u_m + y)\). Then

\[\|u_{m+1} - u_m\| = \|A^{-1}((a - b)u_m + y) - A^{-1}((a - b)u_{m-1} + y)\|\]

\[\leq |A^{-1}||a - b| \|u_m - u_{m-1}\|\]

\[< |A^{-1}|c(|A^{-1}|c)^m\]

by (4)

\[= (|A^{-1}|c)^{m+1}.\]

In addition,

\[\|u_{m+1} - p\| \leq \|u_{m+1} - u_m\| + \|u_m - p\|\]

\[< (|A^{-1}|c)^{m+1} + \sum_{i=1}^m (|A^{-1}|c)^i\]

by the preceding inequality and (4)

\[= \sum_{i=1}^{m+1} (|A^{-1}|c)^i.\]

Finally,

\[\|(a - b)u_{m+1} + y - Ap\| \leq |a - b| \|u_{m+1} - u_m\| + \|(a - b)u_m + y - Ap\|\]

\[< c(|A^{-1}|c)^{m+1} + c \sum_{i=0}^m (|A^{-1}|c)^i\]

\[= c \sum_{i=0}^{m+1} (|A^{-1}|c)^i.\]
This completes the inductive step and defines a sequence $u$ with the following properties:

(7) if $n$ is a positive integer, then

$$
\|u_n - u_{n-1}\| < (|A^{-1}|c)^n \quad \text{and} \quad \|u_n - p\| < \sum_{i=1}^{n} (|A^{-1}|c)^i;
$$

(8) if $n$ is a nonnegative integer, then

$$
\|(a - b)u_n + y - Ap\| < c \sum_{i=0}^{n} (|A^{-1}|c)^i;
$$

(9) $u_0 = p$ and $u_n = A^{-1}((a - b)u_{n-1} + y)$

for each positive integer $n$.

Since $|A^{-1}|c \leq 1/2$, we know that $\sum_{i=0}^{\infty} (|A^{-1}|c)^i$ is convergent and, hence, $u$ is Cauchy. Since $H$ is complete, denote by $x$ the sequential limit of $u$.

To show that $x$ is in $R_p(\delta_b)$ and $(bI - T)x = y$, let us first prove that $(a - b)x + y$ is in $R_{Ap}(\varepsilon_a)$. If $d > 0$ and $t$ is a positive integer such that $\|x - u_t\| < d/c$, we see that

$$
\|(a - b)x + y - Ap\| \leq |a - b|\|x - u_t\| + \|(a - b)u_t + y - Ap\|
$$

$$
< c(d/c) + c \sum_{i=0}^{t} (|A^{-1}|c)^i \quad \text{by (8)}
$$

$$
< d + 2c
$$

$$
\leq d + 2(\varepsilon_a/3) \quad \text{by the choice of $c$.}
$$

Thus $\|(a - b)x + y - Ap\| \leq 2(\varepsilon_a/3) < \varepsilon_a$. This means that, since $A^{-1}$ is continuous by (3), we have

$$
x = \lim_{n \to \infty} u_n = \lim_{n \to \infty} A^{-1}((a - b)u_{n-1} + y)
$$

$$
= A^{-1}\left[\lim_{n \to \infty} ((a - b)u_{n-1} + y)\right] = A^{-1}\left[(a - b)\lim_{n \to \infty} u_{n-1} + y\right]
$$

$$
= A^{-1}((a - b)x + y).
$$

Therefore $Ax = (a - b)x + y$, so $ax - Tx = Ax = (a - b)x + y$ and $(bI - T)x = y$.

To complete the proof it must be shown that $x$ is in $R_p(\delta_b)$. Since $x$ is the sequential limit of $u$, we know by (7) that $\|x - p\| \leq \sum_{i=1}^{\infty} (|A^{-1}|c)^i$, which is $|A^{-1}|c/(1 - |A^{-1}|c)$ by the fact that $|A^{-1}|c < 1$. The fact that

$$
c \leq \frac{\delta_a}{|A^{-1}|(2 + \delta_a)}
$$

implies that $|A^{-1}|c/(1 - |A^{-1}|c) \leq \delta_a/2$. Similarly, the choice of $c$ to be not greater than $r/[|A^{-1}|(2 + r)]$ means that

$$
\|x - p\| \leq |A^{-1}|c/(1 - |A^{-1}|c) \leq r/2.
$$

Thus

$$
\|x - p\| \leq \min\{\delta_a/2, r/2\} < \min\{\delta_a, r\} = \delta_b,
$$
so \( x \) is in \( R_p(\delta_b) \) and \( y \) is in \((bI - T)(R_p(\delta_b))\). This completes the proof.

An example of the phenomenon described by the Theorem is the transformation \( I^* \) on the complex numbers, which maps each complex number onto its conjugate. \( I^* \) is Lipschitzian, hence locally Lipschitzian at each complex number. The local resolvent set of \( I^* \) at 0—indeed at any complex number—is the complement in the complex numbers of the unit circle.

The Theorem reveals a further similarity between the global spectrum of a bounded linear transformation and the local spectrum [2, pp. 212, 213] of certain nonlinear transformations. In doing so it heightens one's hope that there is a suitable local analog for nonlinear transformations to the spectral representation theory for bounded linear ones. One of the next questions to be answered in attempting to discover such a theory seems to be the following: Does a locally Lipschitzian transformation (or a continuously differentiable transformation on a non-finite-dimensional space) have a local spectrum? It is my feeling that the answer is affirmative.

**References**


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