CHARACTERIZATIONS OF URYSOHN-CLOSED SPACES

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Abstract. This paper gives characterizations of Urysohn-closed and minimal Urysohn spaces, some of which make use of nets.

1. Introduction. Our primary interest is the investigation of Urysohn-closed and minimal Urysohn spaces. Characterizations of Urysohn-closed and minimal Urysohn spaces are given in terms of special types of open filterbases [1, p. 101]. Open filterbases, of course, determine nets but not every net determines an open filterbase. We give characterizations of Urysohn-closed and minimal Urysohn spaces in terms of nets and arbitrary filterbases. These characterizations are obtained mainly through the introduction of a type of convergence for filterbases and nets that we call \( u \)-convergence.

Throughout, \( \text{cl}(A) \) will denote the closure of a set \( A \).

2. Preliminary definitions and theorems. Let \( X \) be a topological space and let \( G \) and \( H \) be open sets in \( X \) containing a point \( p \in X \). Then \( G \) and \( H \) will be called an ordered pair of open sets containing \( p \) (denoted by \( (G, H) \)) if \( p \in G \subset \text{cl}(G) \subset H \).

Definition 2.1. Let \( X \) be a topological space and let \( \mathcal{F} = \{ A_\alpha : \alpha \in \Delta \} \) be a filterbase in \( X \). Then \( \mathcal{F} \) \( u \)-converges to \( x \in X \) \((\mathcal{F} \to u x)\) if for each ordered pair of open sets \( (G, H) \) containing \( x \) there exists an \( A_\alpha \in \mathcal{F} \) such that \( A_\alpha \subset \text{cl}(H) \). The filterbase \( \mathcal{F} \) \( u \)-accumulates to \( x \in X \) \((\mathcal{F} \alpha u x)\) if for each ordered pair of open sets \( (G, H) \) containing \( x \) and for each \( A_\alpha \in \mathcal{F} \), \( A_\alpha \cap \text{cl}(H) \neq \emptyset \).

Convergence and accumulation of filterbases in the usual sense, of course, imply \( u \)-convergence and \( u \)-accumulation, respectively. However, the converses do not hold as the next example shows.

Example 2.2. Let \( I = [0, 1] \) have as a subbase the usual open sets together with the set \( A = \{ r : 1/4 < r < 3/4 \text{ and } r \text{ is rational} \} \). Let the filterbase \( \mathcal{F} \) consist of a single element \( B = \{ x : 1/3 < x < 2/3 \text{ and } x \text{ is irrational} \} \) and let \( x = 1/2 \). The filterbase \( \mathcal{F} \) does not converge or accumulate in the usual sense to \( x \) but \( \mathcal{F} \alpha u x \).

There are a number of theorems concerning \( u \)-convergence and \( u \)-accumulation whose statements parallel those of convergence and accumulation in the usual sense. We give a sample of some of these theorems but omit their straightforward proofs.

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Theorem 2.3. In a topological space $X$ the following properties hold:
(a) If $\mathcal{F}$ is a filterbase in $X$ such that $\mathcal{F} \ u$-converges to $x \in X$, then $\mathcal{F} \ u$-accumulates to $x$. If $X$ is a Urysohn space and if $\mathcal{F}$ converges to $x \in X$, then $\mathcal{F} \ u$-accumulates at no point other than $x$.
(b) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filterbases in $X$ where $\mathcal{F}_2$ is stronger than $\mathcal{F}_1$. Then $\mathcal{F}_1 \ u$-accumulates to $x \in X$ if $\mathcal{F}_2 \ u$-accumulates to $x$.
(c) A filterbase $\mathcal{F}_1 \ u$-accumulates to $x \in X$ if and only if there exists a filterbase $\mathcal{F}_2$ stronger than $\mathcal{F}_1$ such that $\mathcal{F}_2 \ u$-converges to $x$.
(d) A maximal filterbase $\mathcal{M}$ in $X \ u$-accumulates to $x \in X$ if and only if $\mathcal{M} \ u$-converges to $x$.

Definition 2.4. Let $X$ be a topological space and let $\mathcal{O} : D \to X$ be a net in $X$. Then $\mathcal{O} \ u$-converges to $x \in X$ (or $\mathcal{O} \to u x$) if for each ordered pair of open sets $(G, H)$ containing $x$, there exists a $b \in D$ such that $\mathcal{O}(T_b) \subseteq \text{cl}(H)$ (where $T_b = \{c \in D : b < c\}$). The net $\mathcal{O} \ u$-accumulates to $x \in X$ (or $\mathcal{O} \cap u x$) if for each ordered pair of open sets $(G, H)$ containing $x$ and for every $b \in D$, $\mathcal{O}(T_b) \cap \text{cl}(H) \neq \emptyset$.

Of course, if $\mathcal{O} : D \to X$ is a net in $X$, the family $\mathcal{O}(\mathcal{O}) = \{\mathcal{O}(T_b) : b \in D\}$ is a filterbase in $X$ and it is routine to verify that:
(a) $\mathcal{F}(\mathcal{O}) \to u X$ if and only if $\mathcal{O} \to u x$.
(b) $\mathcal{F}(\mathcal{O}) \cap u X$ if and only if $\mathcal{O} \cap u x$.

Conversely, every filterbase $\mathcal{F}$ in $X$ determines a net $\mathcal{O} : D \to X$ such that:
(a) $\mathcal{F} \to u x \in X$ if and only if $\mathcal{O} \to u x$.
(b) $\mathcal{F} \cap u x \in X$ if and only if $\mathcal{O} \cap u x$.

The construction of such a net is the same as that of [2, p. 213].

We next state a few theorems concerning $u$-convergence for nets.

Theorem 2.5. In a topological space $X$ the following properties hold:
(a) If $\mathcal{O}$ is a net in $X$ such that $\mathcal{O} \ u$-converges to $x \in X$, then $\mathcal{O} \ u$-accumulates to $x$. If $X$ is a Urysohn space and if $\mathcal{O}$ converges to $x \in X$, then $\mathcal{O} \ u$-accumulates at no point other than $x$.
(b) A net $\mathcal{O} \ u$-accumulates to $x \in X$ if and only if there exists a subnet of $\mathcal{O} \ u$-converging to $x$.
(c) A universal net $\mathcal{O} \ u$-accumulates to $x \in X$ if and only if $\mathcal{O} \ u$-converges to $x$.

3. Filterbases and net characterizations of Urysohn-closed spaces. An open filterbase $\mathcal{F}$ in $X$ is a Urysohn filterbase if and only if for each $p \in A(\mathcal{F})$ (where $A(\mathcal{F})$ denotes the set of accumulation points of $\mathcal{F}$), there is an open neighborhood $U$ of $p$ and some $V \in \mathcal{F}$ such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$ [3]. An open cover $\mathcal{K}$ of a space $X$ is a Urysohn open cover if there exists an open cover $\mathcal{V}$ of $X$ with the property that for each $V \in \mathcal{V}$, there is a $U \in \mathcal{K}$ such that $\text{cl}(V) \subseteq U$. A Urysohn space $X$ is Urysohn-closed provided $X$ is a closed set in every Urysohn space in which it can be embedded [1].

Lemma 3.1. Let $\mathcal{F} = \{O_\alpha : \alpha \in \Delta\}$ be an open Urysohn filterbase on $X$. Then $A(\mathcal{F}) = A_u(\mathcal{F})$ (where $A_u(\mathcal{F})$ denotes the set of $u$-accumulation points of $\mathcal{F}$).

Proof. Clearly we only need to show that $A_u(\mathcal{F}) \subseteq A(\mathcal{F})$. Suppose $p \notin A(\mathcal{F})$. Then there exists an open set $U$ containing $p$ and some $O_\alpha \in \mathcal{F}$ such
that $\text{cl}(U) \cap \text{cl}(O_a) = \emptyset$. The open sets $V = X - \text{cl}(O_a)$ and $U$ form an ordered pair of open sets, $(U, V)$, containing $p$ and have the property that $O_a \cap \text{cl}(V) = \emptyset$. Consequently, $p \not\in A_u(\mathcal{F})$. Therefore, we conclude that $A(\mathcal{F}) = A_u(\mathcal{F})$.

Theorem 4.1 of [1, p. 101] gives several characterizations of Urysohn-closed spaces. We offer the following characterizations.

**Theorem 3.2.** Let $X$ be a Urysohn space. Then the following are equivalent:

(a) $X$ is Urysohn-closed.
(b) Each filterbase $\mathcal{F}$ in $X$ $u$-accumulates to some point $x \in X$.
(c) Each maximal filterbase $\mathcal{M}$ in $X$ $u$-converges to some point $x \in X$.

**Proof.** (a) implies (b). Suppose there exists a filterbase $\mathcal{F} = \{A_a : a \in \Delta\}$ in $X$ that does not $u$-accumulate in $X$. Then for each $x \in X$ there exists an ordered pair of open sets $(U(x), V(x))$ containing $x$ and some $A_a(x) \in \mathcal{F}$ such that $A_a(x) \cap \text{cl}(V(x)) = \emptyset$. Now $\{V(x) : x \in X\}$ is a Urysohn open cover of $X$. Thus by Theorem 4.1 of [1, p. 101], there exists a finite subcollection $\{V(x_i) : i = 1, 2, 3, \ldots, n\}$ such that $\bigcup_{i=1}^{n} \text{cl}(V(x_i)) = X$. Since $\mathcal{F}$ is a filterbase, there exists an $A_{a_0} \subseteq \mathcal{F}$ such that $A_{a_0} \subseteq \bigcap_{i=1}^{n} A_{a(x_i)}$ and $A_{a_0} \neq \emptyset$ implies that for some $j$, $1 \leq j \leq n$, $A_{a_0} \cap \text{cl}(V(x_j)) \neq \emptyset$. Therefore $A_{a(x_j)} \cap \text{cl}(V(x_j)) \neq \emptyset$ which is a contradiction.

(b) implies (a). Let $\mathcal{F} = \{O_a : a \in \Delta\}$ be an open Urysohn filterbase on $X$. By Lemma 3.1 and hypothesis (b) we have that $A_u(\mathcal{F}) = A(\mathcal{F}) \neq \emptyset$. Therefore $X$ is Urysohn-closed according to Theorem 4.1 of [1, p. 101].

(b) implies (c). Let $\mathcal{M}$ be a maximal filterbase in $X$. Then $\mathcal{M}$ $u$-accumulates to some point in $X$ by (b) and hence $u$-converges to that point by Theorem 2.3(d).

(c) implies (b). Let $\mathcal{F}$ be a filterbase in $X$. Then there exists a maximal filterbase $\mathcal{M}$ in $X$ which is stronger than $\mathcal{F}$. Since $\mathcal{M}$ $u$-converges to some point $x \in X$, $\mathcal{F}$ $u$-accumulates to $x$ according to Theorem 2.3.

Our discussion in the previous section showed that filterbases and nets are "equivalent" in the sense of $u$-convergence and $u$-accumulation. Thus we can now characterize Urysohn-closed spaces in terms of nets.

**Theorem 3.3.** In a Urysohn space $X$ the following are equivalent:

(a) $X$ is Urysohn-closed.
(b) Each net in $X$ has a $u$-accumulation point.
(c) Each universal net $u$-converges.

**Remark 3.4.** For each topological space $(X, \tau)$ there is a corresponding topological space $(X, \tau_*)$ called the semiregular space associated with $(X, \tau)$ [1, p. 96]. The topology $\tau_*$ is generated by the regular open sets in $(X, \tau)$. For each open set $U$ in $(X, \tau)$, $\text{cl}(U) = \text{cl}_*(U)$ (where $\text{cl}_*(U)$ denotes the closure of $U$ in $(X, \tau_*)$). Consequently, it follows that a space $(X, \tau)$ is Urysohn if and only if $(X, \tau_*)$ is Urysohn. Also, it is easy to see that a filterbase $\mathcal{F}$ on $X$ $u$-accumulates to $x$ in $(X, \tau)$ if and only if $\mathcal{F}$ $u$-accumulates to $x$ in $(X, \tau_*)$. With this in consideration we give the following theorem.

**Theorem 3.5.** A space $(X, \tau)$ is Urysohn-closed if and only if $(X, \tau_*)$ is Urysohn-closed.

**Proof.** The result follows from Theorem 3.2 and Remark 3.4.
Theorem 4.2 of [1, p. 101] characterizes minimal Urysohn spaces in terms of open Urysohn filterbases. In terms of arbitrary filterbases and \( u \)-convergence, we give the following characterization of minimal Urysohn spaces.

**Theorem 3.6.** Let \( (X, \tau_0) \) be a Urysohn space. Then \( X \) is minimal Urysohn if and only if each filterbase in \( X \) possessing at most one \( u \)-accumulation point is convergent.

**Proof.** Suppose the condition is given and let \( \mathcal{F} \) be an open Urysohn filterbase on \( X \) possessing at most one accumulation point. By Lemma 3.1, \( \mathcal{F} \) possesses at most one \( u \)-accumulation point. Consequently, by hypothesis, \( \mathcal{F} \) converges. This shows that \( X \) is minimal Urysohn according to Theorem 4.2 of [1, p. 101].

Conversely, assume that \( X \) is a minimal Urysohn space and suppose that \( \mathcal{F}_0 = \{ A_\alpha : \alpha \in \Delta \} \) is a filterbase on \( X \) possessing at most one \( u \)-accumulation point. Let \( \mathcal{F} \) be the filter generated by the filterbase \( \mathcal{F}_0 \). Since \( X \) is Urysohn-closed (see Theorem 4.3(a) of [1, p. 101]), \( \mathcal{F}_0 \) has a unique \( u \)-accumulation point \( x \in X \). It follows that the collection of open sets \( \tau_1 = \{ U \in \tau_0 : U \subset X - \{ x \} \} \cup \{ V \in \tau_0 : V \in \mathcal{F} \} \) forms a Urysohn topology on \( X \) with the property that \( \tau_1 \subset \tau_0 \). Suppose there is an open set \( G(x) \in \tau_0 \) containing \( x \) such that for each \( A_\alpha \in \mathcal{F}_0, A_\alpha \not\subset G(x) \). Then for each open \( U(x) \in \tau_1 \) containing \( x, U(x) \not\subset G(x) \) which shows that \( \tau_1 \neq \tau_0 \). Therefore \( (X, \tau_0) \) is not minimal Urysohn, which is a contradiction. We conclude that \( \mathcal{F}_0 \) converges to \( x \).

**Corollary 3.7.** Let \( X \) be a Urysohn space. Then \( X \) is minimal Urysohn if and only if each net in \( X \) possessing at most one \( u \)-accumulation point is convergent.

### 4. First countable Urysohn spaces.

A space \( (X, \tau) \) is called first countable and minimal Urysohn if \( \tau \) is first countable and Urysohn, and if no first countable topology on \( X \) which is strictly weaker than \( \tau \) is Urysohn. \( (X, \tau) \) is first countable and Urysohn-closed if \( \tau \) is first countable and Urysohn, and \( (X, \tau) \) is a closed subspace of every first countable Urysohn space in which it can be embedded.

**Theorem 4.1.** A first countable Urysohn space \( X \) is first countable and Urysohn-closed if each countable filterbase on \( X \) \( u \)-accumulates to some point \( p \in X \).

**Proof.** Let \( \mathcal{F} \) be a countable open Urysohn filterbase on \( X \). By Lemma 3.1, \( A(\mathcal{F}) = A_u(\mathcal{F}) \neq \emptyset \) which implies that \( X \) is first countable and Urysohn-closed according to Theorem 6.3 of [1, p. 107].

**Theorem 4.2.** A first countable Urysohn space \( X \) is first countable and Urysohn-closed if each sequence in \( X \) \( u \)-accumulates to some point \( p \in X \).

**Proof.** Suppose that \( X \) is not Urysohn-closed. Then there exists a first countable Urysohn space \( Y \) and a homeomorphism \( h : X \to h(X) \subset Y \) such that \( h(X) \) is not closed in \( Y \). Thus there exists a point \( p \in Y - h(X) \) (where \( p \in \text{cl}(h(X)) \)) and a sequence \( f : N \to h(X) \), in \( h(X) \) converging to \( p \). Since
$h(X)$ is homeomorphic to $X$, the sequence $f$ $u$-accumulates to some point $z \in h(X)$. Therefore $z = p$ according to Theorem 2.5(a), which is a contradiction.

We say that a point $p \in X$ is a $u$-cluster point of $K \subseteq X$ if for every ordered pair of open sets $(G, H)$ containing $p$, $\text{cl}(H) \cap (K - \{p\}) \neq \emptyset$. We note that in a Urysohn space $X$, a point $p \in X$ is a $u$-cluster point of $K \subseteq X$ if and only if for each ordered pair of open sets $(G, H)$ containing $p$, the closure of $H$ contains infinitely many points of $K$.

**Lemma 4.3.** In a topological space $X$ the following are equivalent:

(a) Every countably infinite subset of $Y$ has at least one $u$-cluster point.

(b) Every sequence in $X$ has a $u$-accumulation point.

**Theorem 4.4.** A first countable Urysohn space $X$ is first countable and Urysohn-closed if every countably infinite subset of $Y$ has at least one $u$-cluster point.

**Proof.** The result follows from Theorem 4.2 and Lemma 4.3.

Theorem 6.3 of [1, p. 107] shows that a first countable Urysohn space $X$ is first countable and minimal Urysohn if every countable open Urysohn filterbase on $X$ with a unique accumulation point is convergent. We show (after Lemma 4.5) that a space $X$ is first countable and minimal Urysohn if each sequence in $X$ with a unique $u$-accumulation point is convergent.

**Lemma 4.5.** If a Urysohn space $X$ has the property that every sequence in $X$ with a unique $u$-accumulation point is convergent, then every sequence in $X$ has a $u$-accumulation point.

**Proof.** Suppose $(x_n)$ is a sequence in $X$ with no $u$-accumulation point. Fix $p \in X$ and define a sequence, $(z_n)$, by $z_n = p$ if $n$ is odd and $z_n = x_{n/2}$ if $n$ is even. It is clear that $p$ is the unique $u$-accumulation point of $(z_n)$ and that $(z_n)$ does not converge to $p$.

**Theorem 4.6.** A first countable Urysohn space $(X, \tau)$ is first countable and minimal Urysohn if every sequence in $X$ with a unique $u$-accumulation point is convergent.

**Proof.** Suppose that $h: (X, \tau) \to (Y, \sigma)$ is a bijective continuous mapping onto a first countable Urysohn space $(Y, \sigma)$. We need to show that $h^{-1}$ is continuous. Let $(y_n)$ be a sequence in $Y$ converging to $y \in Y$. The continuity of $h$ shows that the sequence, $(h^{-1}(y_n))$, has the unique $u$-accumulation point $h^{-1}(y)$. By hypothesis, $(h^{-1}(y_n))$ converges to $h^{-1}(y)$ showing that $h^{-1}$ is continuous.

**References**

