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RECURSIVE EULER AND HAMILTON PATHS

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ABSTRACT. We employ recursion theoretic arguments to show that Hamilton paths for locally finite graphs are more difficult to find, in general, than Euler paths. A locally finite graph is recursive if we can effectively decide whether or not any two given vertices are adjacent, and highly recursive if we can effectively find all vertices adjacent to any given vertex. We find that there are recursive planar graphs with Euler or Hamilton paths but no such recursive paths. There are even particularly simple classes of connected, planar, highly recursive graphs for which we can show there is no effective way to decide about the existence of Euler or Hamilton paths. However, we obtain the following contrast: If a highly recursive graph has an Euler path we can effectively find a recursive Euler path; whereas, there is a planar, highly recursive graph with Hamilton paths but no recursive Hamilton path.

1. **Introduction.** The complete characterization of finite directed or undirected graphs having Euler paths is well known, and characterizations in the infinite case are in the literature (see Ore [9] and Nash-Williams [8]). However, no analogous characterizations for Hamilton paths have been obtained. Our goal here is to find a contrast in the difficulty of finding Euler vs. Hamilton paths in a recursion theoretic setting to reflect the apparent differences in difficulty of these characterization problems. Related applications of recursion theory to combinatorial problems may be found in Jockusch [2], Manaster and Rosenstein [5], [6], and Bean [1].

In the following we will consider a *graph* G to be a symmetric, locally finite, binary relation on N, the natural numbers. Think of N as the set of *vertices* for the graph and G as the set of *edges*. We will use 'vertex' and 'number' interchangeably as context permits. Two vertices (edges) sharing a common edge (vertex) are *adjacent*. G is *planar* if it has a planar representation. The *degree* of a vertex is the number of vertices adjacent to it.

A *one-way Euler* (*Hamilton*) *path* for G is a one-to-one enumeration of all edges (vertices) of G with consecutive edges (vertices) adjacent. A *two-way* or *endless Euler* (*Hamilton*) *path* for G is a one-to-one correspondence between Z, the integers, and the set of all edges (vertices) of G with consecutive edges (vertices) adjacent. In the case of Euler paths we further stipulate that no three consecutive edges share a common vertex unless one edge is a loop.

ϕ_e denotes the partial recursive function with *index* e and ϕ_e^n denotes ϕ_e restricted to n steps for each computation. $\langle\ \rangle$: {finite sequences over

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$N\} \to N$ is a fixed recursive coding of all finite sequences over N so that the sequence number $\langle a_1, a_2, \ldots, a_n\rangle$ codes the sequence $a_1, a_2, \ldots, a_n$. We assume a standard indexing of the partial recursive functions so that $\{(e,x)$: $\phi_e(x)$ is defined$\}$ can be effectively enumerated.

Additional terminology and notation will generally follow Ore [9] in graph theory and Rogers [10] in recursive function theory. An excellent brief introduction to recursive function theory in Manaster and Rosenstein [5] will give the unfamiliar reader sufficient background to follow this work (as well as any of the above-mentioned papers).

In §2 we consider Euler and Hamilton paths for recursive graphs and in §3 we consider a stronger notion of effectiveness. §4 extends all results to directed graphs and in §5 we consider disjoint path decompositions of graphs.

2. **Recursive graphs.** For any of the four possible types of paths considered here, one-way or endless, Euler or Hamilton, we have

THEOREM 1. *There is a recursive planar graph G with such a path but no such recursive path.*

PROOF. We will give the construction in detail for the one-way Euler case and then indicate the modifications necessary for the other cases. The construction of G will proceed in stages.

At stage $n = 2^e q$, q odd, adjoin the least numbered vertex v_n not yet used to v_{n-1} ($n \geqslant 0$; $v_0 = 0$). Go to stage $n + 1$ unless $\langle v_e, v_{e+1}\rangle$ and $\langle v_{e+1}, v_{e+2}\rangle$ occur consecutively in the range of ϕ_e after n steps in the enumeration of $\{(x,y)$: $\phi_e(x) = y\}$. In this case adjoin $v_n + 1$ to v_{e+1} and $v_n + 2$, and adjoin $v_n + 2$ to v_{e+1}. Go to stage $n + 1$.

Trivially G is planar and has Euler paths. G is recursive because previously introduced vertices are never joined to each other at later stages of the construction. Given vertices i and j, to find out if they are adjacent simply carry out the construction of G to a stage in which they both appear. If they are not adjacent at this stage they never will be.

Finally, any purported Euler path, ϕ_e, which is recursive, must eventually commit itself and enumerate edges (v_e, v_{e+1}) and (v_{e+1}, v_{e+2}) consecutively; but then at the next appropriate stage n an edge $(v_n + 1, v_n + 2)$ appears which ϕ_e cannot enumerate correctly. Therefore G has no recursive Euler path.

To obtain the same result for Hamilton paths alter the construction as follows: If v_e and v_{e+1} occur consecutively in the range of ϕ_e after n steps in the enumeration of $\{(x,y)$: $\phi_e(x) = y\}$ adjoin $v_n + 1$ to v_e and v_{e+1}. (Note that in this case G will have a unique but not recursive Hamilton path.)

For endless paths make the obvious modifications of the above constructions using two-way infinite chains.

Theorem 1 essentially shows that information about recursive graphs is too limited for our purposes. In the next section we consider another notion of effectiveness which will prove sufficiently strong.

3. **Highly recursive graphs.** A graph is *highly recursive* if there is a recursive function f: $N \to \{$sequence numbers$\}$ such that $f(i) = \langle i_1, i_2, \ldots, i_n\rangle$ means vertex i is adjacent to exactly the vertices $i_1, i_2, \ldots, i_n$. Note that we have

complete local information about highly recursive graphs. Even so, the following lemma will show that there is no hope of finding out whether or not there is an Euler or Hamilton path for a particularly simple class of highly recursive graphs.

LEMMA 1. *Suppose there is a number $a \geqslant 0$ and recursive functions b and c such that, for all n, ϕ_a, $\phi_{b(n)}$ and $\phi_{c(n)}$ agree on the domain $\{0,1,\ldots,n\}$. Then there is no number d such that, for all n, ϕ_d is 0-1 valued on all the indices of the functions ϕ_a, $\phi_{b(n)}$, and $\phi_{c(n)}$, 0-valued on all the indices of the functions $\phi_{b(n)}$, and 1-valued on all the indices of the functions $\phi_{c(n)}$.*

PROOF. Assume there is such a d. Define a recursive f as follows:

$$\phi_{f(e)}(n) = \begin{cases} \phi_{b(k)}(n) & \text{if } k \leqslant n \text{ is least such that } \phi_d^k(e) = 1, \\ \phi_{c(k)}(n) & \text{if } k \leqslant n \text{ is least such that } \phi_d^k(e) = 0, \\ \phi_a(n) & \text{else.} \end{cases}$$

Thus

$$\phi_{f(e)} = \begin{cases} \phi_{b(k)} & \text{for some } k, \text{ if } \phi_d(e) = 1, \\ \phi_{c(k)}, & \text{for some } k, \text{ if } \phi_d(e) = 0, \\ \phi_a & \text{else.} \end{cases}$$

By the recursion theorem [10, p. 180] there is an x such that $\phi_x = \phi_{f(x)}$. Thus $\phi_d(x) = 0$ or 1 and in fact $\phi_x = \phi_{b(k)}$, for some k, if $\phi_d(x) = 1$ and $\phi_x = \phi_{c(k)}$, for some k, if $\phi_d(x) = 0$. Either way we obtain a contradiction.

Now let a code a set of instructions to generate the graph in Figure 1a (i.e. for every x, $\phi_a(x)$ gives the vertices adjacent to vertex x), let $b(n)$ code a set of instructions to generate a graph with a circuit of length $2n + 3$, as represented in Figure 1b, and let $c(n) = a$. Clearly any procedure to find out whether or not a graph has a one-way Euler or Hamilton path will produce a d contradicting Lemma 1. A similar argument with Figures 1c, 1d (1e, 1f) shows that there is no effective way to decide whether or not a graph has an endless Euler (Hamilton) path.

However,

THEOREM 2. *There is an effective procedure to find a recursive one-way Euler path for any highly recursive graph which has a one-way Euler path.*

PROOF. The proof is a straightforward effective version of Ore's proof [9, p. 43] of the following characterization of graphs with one-way Euler paths:

(*) A connected graph G has a one-way Euler path if and only if (i) it has exactly one vertex of odd degree and (ii) if H is any finite subgraph of G then $G - H$ has exactly one infinite connected component.

Note that the following procedures are effective for highly recursive graphs:

(1) Find a given edge;
(2) Find the degree of a given vertex;
(3) Give a one-to-one enumeration $\lambda_0, \lambda_1, \lambda_2, \cdots$ of all edges;

(4) Find a path between a vertex and an edge or another vertex if such a path exists;

(5) Given that a graph has k infinite components and given $l > k$ specific vertices, categorize the l vertices into at most k potentially infinite components (and possibly some finite components as well).

Recall that:

(6) No finite graph has an odd number of odd degree vertices.

Also recall Euler's result that:

(7) A finite connected graph has an Euler path if and only if it has either no odd degree vertices or two odd degree vertices. In the former case the path can start at any vertex and must end at the same vertex. In the latter case it must start at one odd vertex and end at the other.

CONSTRUCTION. Given a highly recursive graph G satisfying the conditions of (*), proceed in stages. At stage 0 locate the odd degree vertex (procedure 2) and edge λ_0 (procedure 1) and find a path P connecting them and including λ_0 (procedure 4). Find the vertices of P which are in the finite components of $G - P$ (procedure 5 with $k = 1$). Note that all vertices in $G - P$ except p_0, the other end of P, have even degree. Therefore p_0 must be in the infinite component (by (6)) and P may be expanded to a path P' from a to p_0 which includes all finite components of $G - P$ (by (7)) so that $G - P'$ is connected and satisfies the conditions of (*). Let $P_0 = P'$.

At stage n assume P_{n-1}, a path from a to p_{n-1}, has been defined. Let λ_k be the least edge in $G - P_{n-1}$. Repeat stage 0 above with $G - P_{n-1}$, p_{n-1}, λ_k, and p_n in place of G, a, λ_0, and p_0 respectively. Extend P_{n-1} with the new P' to obtain P_n, a path from a to p_n. Go to stage $n + 1$.

Obviously the P_n for $n = 0,1,2, \cdots$ can be used to construct the desired recursive enumeration of adjacent edges.

THEOREM 3. *There is an effective procedure to find a recursive endless Euler path for any highly recursive graph which has an endless Euler path.*

PROOF. The proof is a more involved effective version of Ore's proof [9, p. 45] of the following characterization of graphs with endless Euler paths:

(**) A connected graph G has an endless Euler path if and only if (i) it has no odd degree vertices, (ii) if H is a finite subgraph then $G - H$ has at most two infinite components, and (iii) if H has no odd degree vertices then $G - H$ has exactly one infinite component.

Since there is no effective way to find out if $G - H$ has one or two infinite components if condition (iii) does not apply, the real problem in modifying Ore's proof is to construct an endless Euler path without this knowledge. We will need two lemmas.

LEMMA 2. *Let P be a path from a to b in a graph G with an endless Euler path. There is an effective procedure which will expand P to* (i) *a circuit C' such that $G - C'$ is connected and shares at least one vertex with C' or* (ii) *a path P' such that $G - P'$ has no finite component, each of a,b is in an infinite component, and each infinite component contains at least one of a,b.*

PROOF. Apply procedure 5 with $k = 2$ to the vertices of P in $G - P$. If a

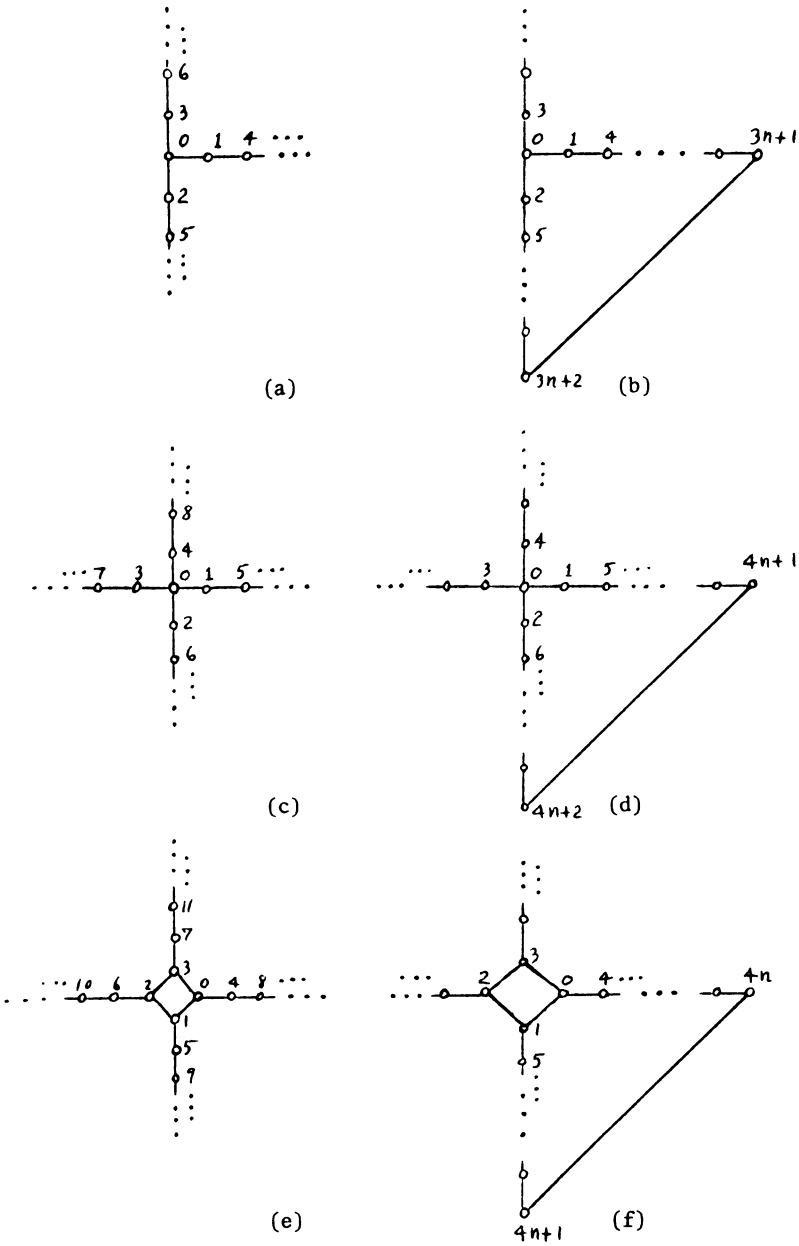

FIGURE 1

and b are eventually found to be in the same component then expand P to a circuit C and apply procedure 5 with $k = 1$ to the vertices of C in $G - C$. Since all vertices of $G - C$ are of even degree C may be expanded to a circuit C' including all finite components (using (7)). Also, by condition (iii) of (**), $G - C'$ has only one component, which must share a vertex with C' since G was connected.

If, when all vertices of P in $G - P$ have been merged into finite and two possibly infinite components, a and b have not been connected, then the components of a and b are the only possible infinite ones since a and b are of odd degree in $G - P$ (using (6)). Again using (7), expand P to P' so that $G - P'$ has no finite component.

LEMMA 3. *If case* (ii) *of Lemma* 2 *holds and Q is a path from a to c in $G - P'$ then there is an effective procedure to expand Q to* (i) *a path Q' from a to b such that $G - Q'$ has one component, sharing at least one vertex with Q' or* (ii) *a path Q'' such that $G - Q''$ has no finite component, each of b,c is in an infinite component, and each infinite component contains at least one of b,c.*

PROOF. Apply Lemma 2 to the path $P' + Q$. By the conditions of Lemma 2, case (ii), a vertex shared with $G - P' - Q$ can be chosen from Q'.

CONSTRUCTION. Given a highly recursive graph satisfying the conditions of (**):

Stage 0. Find a path from vertex 0 including and ending at λ_0. Apply Lemma 2 with $a = 0$ and b the other endpoint. If case (i) applies let $p_0 = p_1$ be a vertex common to C' and $G - C'$. Trace C' using two disjoint paths P_0 and P_1 starting at $a = 0$ and ending at $p_0 = p_1$. Note that $G - P_0 - P_1$ satisfies the conditions of (**). If case (ii) applies let $p_1 = a$ and $p_0 = b$. Let $P_0 = P'$ and let P_1 be the null path from p_1. Note that $G - P_0 - P_1$ has the properties of $G - P'$ in case (ii) of Lemma 2.

Stage $n + 1$. Assume P_{2n} and P_{2n+1}, disjoint paths from 0 to p_{2n}, p_{2n+1} respectively, are defined.

(I) If $p_{2n} = p_{2n+1}$ and $G - P_{2n} - P_{2n+1}$ satisfies the conditions of (**) find the least λ_k in $G - P_{2n} - P_{2n+1}$ and a path from p_{2n} including and ending at λ_k. Apply Lemma 2 with $a = p_{2n}$.

If case (i) applies let $p_{2n+2} = p_{2n+3}$ be a vertex in common with C' and $G - P_{2n} - P_{2n+1} - C'$. Extend P_{2n} to P_{2n+2} and P_{2n+1} to P_{2n+3} by tracing C' with two disjoint paths starting at a and ending at $p_{2n+2} = p_{2n+3}$. Note that the conditions of (**) are met by $G - P_{2n+2} - P_{2n+3}$.

If case (ii) applies extend P_{2n} to P_{2n+2} by concatenating P', let the endpoint be p_{2n+2} and let $P_{2n+3} = P_{2n+1}$, $p_{2n+3} = p_{2n+1}$. Note that $G - P_{2n+2} - P_{2n+3}$ has the properties of $G - P'$ in Lemma 2, case (ii).

(II) If $p_{2n} \neq p_{2n+1}$ and $G - P_{2n} - P_{2n+1}$ has the properties of $G - P'$ in case (ii) of Lemma 2, find the least λ_k in $G - P_{2n} - P_{2n+1}$ and simultaneously try to find a path from p_{2n} and from p_{2n+1} ending in and including λ_k. When a path is found, say from p_{2n}, apply Lemma 3 with $a = p_{2n}$ and $b = p_{2n+1}$.

If case (i) applies let $p_{2n+2} = p_{2n+3}$ be a vertex of Q' common to $G - P_{2n} - P_{2n+1} - Q'$. Extend P_{2n} to P_{2n+2} and P_{2n+1} to P_{2n+3}, both ending at $p_{2n+2} = p_{2n+3}$ and (disjointly) including Q'. Observe that the conditions of (**) apply to $G - P_{2n+2} - P_{2n+3}$.

If case (ii) applies extend P_{2n} to P_{2n+2} with endpoint p_{2n+2} by concatenating Q''. Let $P_{2n+3} = P_{2n+1}$ and $p_{2n+3} = p_{2n+1}$. Observe that $G - P_{2n+2} - P_{2n+3}$ has the properties of $G - P'$ in case (ii) of Lemma 2.

Go to stage $n + 2$.

Obviously the P_{2n} and P_{2n+1} for $n = 0,1,2, \cdots$ can be used to construct a recursive correspondence with Z of consecutively adjacent edges. (The path

must be endless because G has no one-way Euler path.)

In contrast to Theorems 2 and 3 we have

THEOREM 4. *There is no effective procedure which, given a highly recursive planar graph with one-way (endless) Hamilton path, will find a recursive one-way (endless) Hamilton path.*

PROOF. Let a code a set of instructions to generate the graph in Figure 2a, and let $b(n)$ and $c(n)$ code sets of instructions to generate graphs as represented in Figures 2b and 2c, respectively, with ladders of height $2n$. Note that the graphs represented in Figure 2b (2c) have a unique Hamilton path $0,1,3,2,4,5 \cdots$ $(0,1,2,3,5,4 \cdots)$. A procedure to find a one-way Hamilton path would then produce a d contradicting Lemma 1. For endless Hamilton paths use the graphs represented in Figures 2d, 2e and 2f.

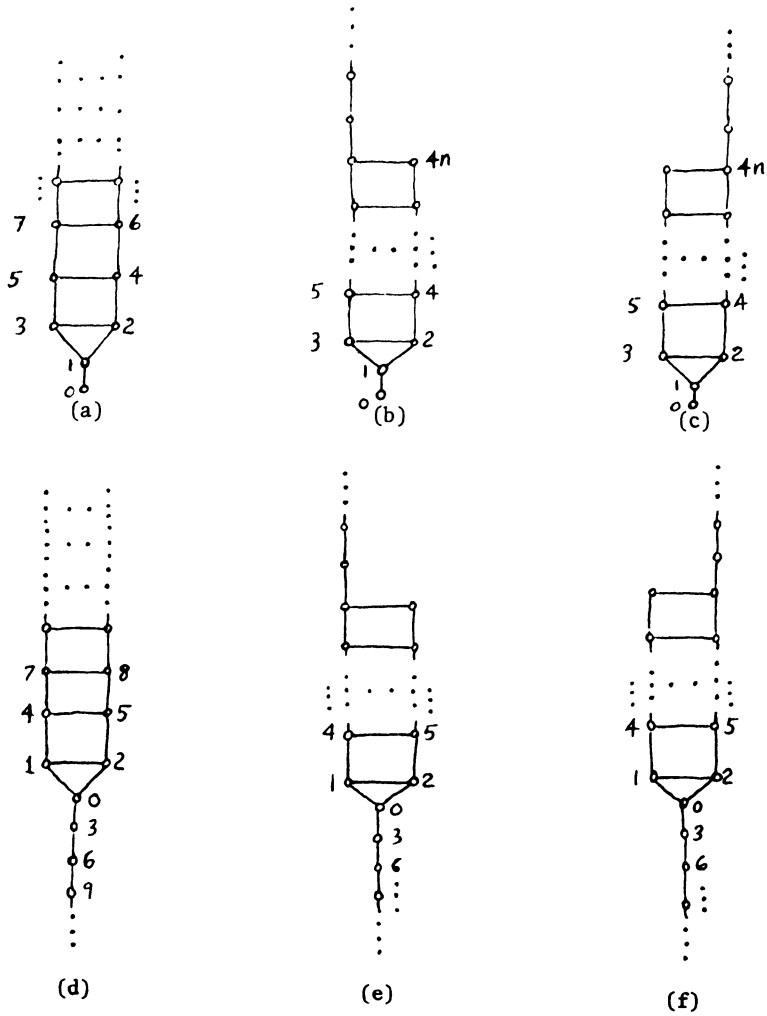

FIGURE 2

We remark that Theorems 2 and 3 can be regarded as constructive versions of (*) and (**) while Theorem 4 can be reformulated to show there is no hope for constructive analogs of (*) and (**) for Hamilton paths.

We can strengthen Theorem 4 using the correspondence given in

THEOREM 5. *For every highly recursive tree T there is a highly recursive planar graph G and a recursive isomorphism between paths through T and Hamilton paths for G. (T, in the terminology of Ore* [9], *is a locally finite, rooted tree and a 'path through T' is an infinite path starting at the root.)*

PROOF. Given T we construct G in stages. At stage 0 introduce vertices labeled 0 and 0^{out} and edge $(0, 0^{\text{out}})$ (corresponding to the root node of T).

At stage n, if $\sigma_0, \sigma_1, \ldots, \sigma_m$ are the nodes of the tree at level n (i.e. on a finite path of length n from the root) introduce a circuit of the least $3(m+1)$ new vertices to G, surrounding the part of G defined at stage $n-1$ and labeled in the order shown in the circuit representation

$$\left(\sigma_0^{\text{out}}, \sigma_0^{\text{in}}, \sigma_{0,1}, \sigma_1^{\text{out}}, \sigma_1^{\text{in}}, \sigma_{1,2}, \ldots, \sigma_m^{\text{out}}, \sigma_m^{\text{in}}, \sigma_{m,0}, \sigma_0^{\text{out}}\right).$$

For every τ_i at level $n-1$ and σ_j at level n adjacent to τ_i in T also add an edge $(\tau_i^{\text{out}}, \sigma_j^{\text{in}})$ to G. (For the endless case also add a vertex v_n and edge (v_{n-1}, v_n), where $v_0 = 0$.)

Go to stage $n+1$.

From the construction G is clearly a highly recursive planar graph. We will show that the desired correspondence between a tree path and Hamilton path is given by σ_j immediately follows τ_i on the tree path if and only if σ_j^{in} immediately follows τ_i^{out} on the Hamilton path.

It suffices to show by induction on n that a Hamilton path for G which enters the stage n circuit at σ_j^{in} must leave the circuit only at σ_j^{out}. The base step is trivial. Suppose all Hamilton paths for G have this property out to stage $n-1$ of G and consider a Hamilton path which enters the stage n circuit at σ_j^{in}. The path must then proceed to $\sigma_{j,k}$ ($k \equiv j+1 \pmod{m+1}$) or else it would be trapped when it finally came to $\sigma_{j,k}$. Similarly, the path cannot leave the circuit at any σ_l^{out}, $l \neq j$, or else it would be trapped when it finally got to σ_l^{in}. (There is of course no problem if $l = j$ since σ_j^{in} is already on the path.) Q.E.D.

Theorem 5 yields numerous corollaries to work of Jockusch and Soare [3], [4]. Among the more elementary is the following, which clearly strengthens Theorem 4:

COROLLARY. *There is a highly recursive planar graph with one-way (endless) Hamilton paths but no recursive Hamilton path.*

PROOF. Given a disjoint pair of recursively enumerable sets, A and B, we define a highly recursive binary tree in stages, as we generate A and B. At stage n we stop the growth (at level n) of any path which turns left (right) at level i if we discover at this stage that i is in A (B). Obviously the paths through this tree can be identified with the characteristic functions of sets C which contain A and are disjoint from B. Since A and B can be chosen to be recursively inseparable (see Rogers [10, p. 170]) there is a highly recursive tree with paths, but none recursive. The Corollary now follows from Theorem 5.

4. **Directed graphs.** We can relax the symmetry restriction on graphs, obtaining directed graphs (digraphs). Euler and Hamilton dipaths can be defined in the obvious way and all results above extend to digraphs using obvious modifications, including Theorems 2 and 3, since the more complicated characterization conditions given by Nash-Williams [8] are appropriately inherited by the complements of any finite subpaths of the Euler dipaths being constructed. Theorem 5 goes over by replacing each edge in the graph constructed by two directed edges with opposite orientations.

5. **Disjoint path decompositions.** In the literature there are extensions of (**) to characterizations of graphs which admit a decomposition into disjoint endless paths (see Nash-Williams [7]). Theorem 3 does not extend in a corresponding natural way, however, as shown by

THEOREM 6. *There is no effective procedure to decompose a highly recursive planar graph into two disjoint recursive endless paths even given that there is such a decomposition.*

PROOF. Let a code a set of instructions to generate the graph in Figure 3a and let $b(n)$ code a set of instructions to generate a graph as represented in Figure 3b, with the break occurring after a chain of n vertices from vertex 0 as shown. Let $c(n) = a$. Clearly any decomposition procedure will produce a d contradicting Lemma 1.

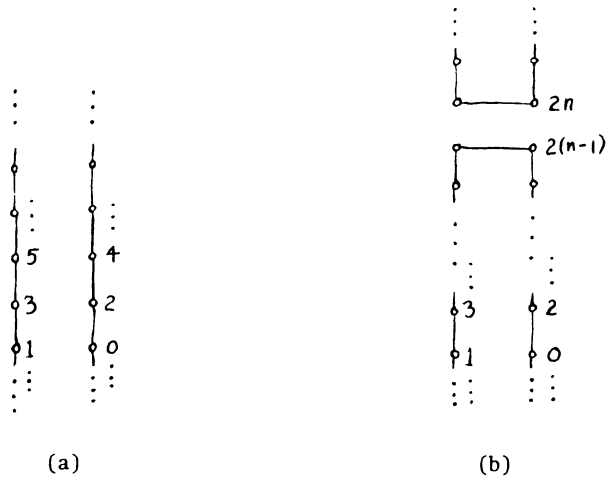

FIGURE 3

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