SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON NOWHERE MONOTONE FUNCTIONS

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Abstract. The existence of everywhere differentiable but nowhere monotone functions is established using the Baire Category Theorem, and the relatively easy fact that there are nontrivial bounded derivatives with a dense set of zeros.

Interest in everywhere differentiable, nowhere monotone functions was revived by Katznelson and Stromberg in [4] where they gave a construction of such a function which is considerably simpler than the original one due to Köpcke or the one in the book by Hobson [3, pp. 412–421]. This work was followed up by Goffman in [2] where a much shorter construction is given but which uses a deep theorem due to Zahorski. Here the existence of such functions is established using the Baire Category Theorem.

Let $R$ denote the real line and let

$$D = \{ f: R \rightarrow R: f \text{ is bounded and there is a function } F \text{ such that } F'(x) = f(x) \text{ for all } x \in R \},$$

and endow $D$ with the metric

$$d(f, g) = \sup_{x \in R} |f(x) - g(x)|.$$

This is the metric of uniform convergence, and by a standard advanced calculus theorem, a uniform limit of a sequence of bounded derivatives is a bounded derivative. Hence $D$ is a complete metric space. Let

$$D_0 = \{ f \in D: \{ x: f(x) = 0 \} \text{ is dense in } R \},$$

and give to $D_0$ the metric of $D$. Then $D_0$ itself is complete for if $\{ f_k \}$ is a sequence in $D_0$ converging in metric to $f \in D$, then for each $k$, $Z_k = \{ x: f_k(x) = 0 \}$ is a dense $G_δ$ set and hence $Z = \bigcap_{k=1}^{\infty} Z_k$ is dense in $R$. But $Z \subset \{ x: f(x) = 0 \}$. Thus $f \in D_0$.

It is not hard to show that $D_0$ contains more than just the zero function (see [1, p. 27] or [5]). The existence of such a function and the fact that $D_0$ is closed

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under addition will be used below. The proof of the latter is like the completeness of $D_0$ only easier.

**Theorem.** Let

$$ E = \{ f \in D_0 : \text{there is an interval on which } f \text{ is unsigned} \}. $$

Then $E$ is of the first category in $D_0$.

**Proof.** Let $\{I_n\}$ be an ordering of the collection of all closed intervals having rational endpoints. Let

$$ E_n = \{ f \in D_0 : f(x) \geq 0 \text{ for all } x \in I_n \} $$

and

$$ F_n = \{ f \in D_0 : f(x) \leq 0 \text{ for all } x \in I_n \}. $$

Then clearly

$$ E = \bigcup_{n=1}^{\infty} (E_n \cup F_n); $$

so it suffices to prove that $E_n$ and $F_n$ are closed and contain no spheres. The argument will be carried out for $E_n$. A similar procedure works for $F_n$.

That $E_n$ is closed is immediate. To prove that $E_n$ contains no sphere suppose $f \in D_0$ and $\varepsilon > 0$. Since $f \in D_0$ there is an $x \in I_n$ such that $f(x) = 0$. Since there are bounded derivatives having a dense set of zeros that are not identically zero, by pushing and crushing it is not hard to prove that there is a function $h \in D_0$ such that $h(x) < 0$ and $\sup_{y \in R} |h(y)| < \varepsilon$. Then $g = f + h$ belongs to $D_0$, $d(f,g) < \varepsilon$, and $g \notin E_n$ since $g(x) = f(x) + h(x) = h(x) < 0$ and $x \in I_n$. Thus the sphere of radius $\varepsilon$ about $f$ is not contained in $E_n$.

**References**


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