ON LARGE CYCLIC SUBGROUPS OF FINITE GROUPS

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Abstract. It is known that for each (composite) $n$ every group of order $n$ contains a proper subgroup of order greater than $n^{1/3}$. We prove that given $0 < \delta < 1$, for almost all $n \leq x$, as $x \to \infty$, every group $G$ of order $n$ contains a characteristic cyclic subgroup of square-free order $> n^{1-1/((\log n)^{1-\delta}}$, and provide an upper bound to the number of exceptional $n$. This leads immediately to a like density result for a lower bound to the number of conjugacy classes in $G$.

From the deep theorem by Feit and Thompson [6] that all groups of odd order are solvable, it immediately follows that for every odd (composite) integer $n$, if $G$ is a group of order $n$ then $G$ contains a proper subgroup of order $> n^{1/2}$. On the other hand, Brauer and Fowler [2] showed that every group $G$ of even order $n > 2$ contains a proper subgroup of order $> n^{1/3}$.

Denoting by $k(G)$ the number of conjugacy classes in the finite group $G$, we know that for every $n$, $k(G) > \log_2 \log_2 n$ if $G$ has order $n$ (see, e.g., [5] or [8]). Recently [1] the author showed that given any $c < \log 2$, for almost all integers $n \leq x$, as $x \to \infty$, $k(G) > (\log n)^c$ for each $G$ of order $n$. Here we give a proof of the following

Theorem. Given $0 < \delta < 1$, almost all integers $n \leq x$, as $x \to \infty$, have the property that every group of order $n$ contains a characteristic cyclic subgroup of square-free order $> n^{1-1/((\log n)^{1-\delta}}$, where the number of exceptional integers is $< x(2 \log \log x)/(\log x)^\delta$ for all large $x$.

As an immediate corollary we considerably improve the above density result on the lower bound for $k(G)$, now obtaining $k(G) > n^{1-c}$.

Finally, we note that Erdös [4], sharpening the results of Dornhoff and Spitznagel [3] on the scarcity of simple group orders, proved that for almost all $n \leq x$, every group of order $n$ has a normal Sylow $p$-subgroup, where $p$ is the largest prime factor of $n$, and the number of exceptional integers is

$$< x/\exp[(1/\sqrt{2} + O(1))(\log x \log \log x)^{1/2}].$$

In the course of the proof of our theorem we find that if $\{\epsilon_n\}$ is a sequence tending to 0 (however slowly) then for almost all $n \leq x$, as $x \to \infty$, every group of order $n$ has a normal Sylow $p$-subgroup of prime order $p > n^{\epsilon_n}$.

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where of course the number of exceptional integers has an upper bound depending on \( \{e_n\} \).

**Lemma 1.** The number of positive integers \( n \leq x \), such that \( p^2 \mid n \) for some prime \( p > f(x) \), is less than \( x/f(x) \).

**Proof.** Since, for fixed \( p \), the number of integers \( \leq x \) which are divisible by \( p^2 \) is \( \lfloor x/p^2 \rfloor \), the number sought in the lemma is certainly no more than

\[
\sum_{p > f(x)} \left\lfloor \frac{x}{p^2} \right\rfloor \leq x \sum_{p > f(x)} \frac{1}{p^2} < x \sum_{m > f(x)} \frac{1}{m^2} < x \int_{f(x)}^{\infty} \frac{dt}{t^2} = \frac{x}{f(x)}.
\]

**Lemma 2.** The number of positive integers \( \leq x \) with a prime factor \( p > f(x) \), and simultaneously a divisor \( d > 1 \) satisfying \( d \equiv 1 \pmod{p} \), is less than \( x(\log x + 1)/f(x) \).

**Proof.** For fixed \( p \), the number of positive integers \( \leq x \), which are simultaneously divisible by \( p \) and some divisor \( d > 1 \) satisfying \( d \equiv 1 \pmod{p} \), is at most \( \sum_{p^m \mid x} \left\lfloor \frac{x}{p^m(p+1)} \right\rfloor \). Thus, the number sought in the lemma is no more than

\[
\sum_{p > f(x)} \left\lfloor \frac{x}{p^2} \right\rfloor \left( \sum_{i=1}^{[x/p^2]} \frac{1}{i} \right) < x \left( \sum_{m > f(x)} \frac{1}{m^2} \right) < x(\log x + 1)/f(x).
\]

**Lemma 3.** The number of integers \( \leq x \) which have a divisor \( d \geq h(x) \), such that each prime factor of \( d \) is \( \leq g(x) \), is less than \( x(\log(g(x)) + c_1)/\log(h(x)) \).

**Proof.** If \( m_1, m_2, m_3, \ldots, m_N \) denote these integers, then in \( \prod_{i=1}^{N} m_i \) the contribution of the primes \( \leq g(x) \) is at least \( h^N(x) \). On the other hand, the primes \( \leq g(x) \) certainly contribute no more to \( \prod_{i=1}^{N} m_i \) than their contribution to \( [x]! \) Hence

\[
h^N(x) \leq \prod_{p \leq g(x)} p^{[\log(p)/\log(p-1)]} = \prod_{p \leq g(x)} \frac{p^{(\log(p)/\log(p-1))}}{p^{(\log(p)/\log(p-1))}}
\]

or

\[
\frac{N \log(h(x))}{x} < \sum_{p \leq g(x)} \frac{\log p}{p - 1} = \sum_{p \leq g(x)} \frac{\log p}{p} + \sum_{p \leq g(x)} \frac{\log p}{p(p - 1)} < \sum_{p \leq g(x)} \frac{\log p}{p} + \sum_{j=2}^{\infty} \frac{\log j}{j(j - 1)} < \log(g(x)) + c_1
\]

since we know [7, 22.6] that

\[
\sum_{p \leq g(x)} \frac{\log p}{p} = \log(g(x)) + O(1),
\]

and the infinite sum converges.
Lemma 4. Given $0 < \delta < 1$, almost all integers $n \leq x$, as $x \to \infty$, have a square-free divisor $n_0$ with the properties:

(i) if a prime $p$ divides $n_0$, then $p > (\log x)^{1+\delta}$;
(ii) for each prime $p$ which divides $n_0$, if $d > 1$ divides $n$, then $d \not\equiv 1 \pmod{p}$;
(iii) $(n_0, n/n_0) = 1$;
(iv) $n_0 > n^{1-1/(\log n)^{1-\delta}}$.

Proof. Given $0 < \delta < 1$, almost all $n \leq x$ satisfy $n > x^\delta$; for such $n$ and all large $x$,

$$\frac{n}{\exp((\log x)^{\delta})} > n^{1-1/\delta (\log x)^{1-\delta}} > n^{1-1/(\log n)^{1-\delta}}.$$ 

Lemma 3, with $g(x) = (\log x)^{1+\delta}$ and $h(x) = \exp((\log x)^{\delta})$, implies that the number of integers $n \leq x$, with prime decomposition

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l} p_{l+1}^{\alpha_{l+1}} \cdots p_r^{\alpha_r}(p_i < p_{i+1})$$

satisfying

$$p_i < (\log x)^{1+\delta} < p_{i+1} \quad \text{and} \quad \prod_{i=1}^l p_i^{\alpha_i} \geq \exp((\log x)^{\delta})$$

for some $l$, is less than $x((1+\delta)\log \log x + c_1)/(\log x)^{\delta}$ (showing, since we may assume $x^\delta < n$, that almost all integers $n \leq x$, as $x \to \infty$, have a prime factor $> (\log x)^{1+\delta}$). Lemmas 1 and 2 (with $f(x) = (\log x)^{1+\delta}$) now show that except for at most $x/(\log x)^{1+\delta} + x(\log x + 1)/(\log x)^{1+\delta}$ of those integers $n \leq x$, $n = \prod_{i=1}^l p_i^{\alpha_i} \prod_{j=l+1}^r p_j^{\alpha_j}$, $p_i < (\log x)^{1+\delta} < p_{i+1}$ and $\prod_{i=1}^l p_i^{\alpha_i} \leq \exp((\log x)^{\delta})$, we have for $l + 1 < j < \nu(n)$: (a) $\alpha_j = 0$ and (b) $d | n$ and $d > 1 \Rightarrow d \not\equiv 1 \pmod{p_j}$. For such $n$, put $n_0 = \prod_{j=l+1}^r p_j$. Then $n_0$ is square-free and satisfies (i) through (iv). Finally, the number of integers $n \leq x$ which do not have such a square-free divisor $n_0$ is less than

$$x^\delta + \frac{x((1+\delta)\log \log x + c_1)}{(\log x)^{\delta}} + \frac{x(\log x + 1)}{(\log x)^{1+\delta}} \leq 2x \frac{\log \log x}{(\log x)^{\delta}} \quad \text{for all large } x.$$

Theorem. Given $0 < \delta < 1$, almost all integers $n \leq x$, as $x \to \infty$, have the property that every group of order $n$ has a characteristic cyclic subgroup of square-free order $n_0 > n^{1-1/(\log n)^{1-\delta}}$, where $(n_0, n/n_0) = 1$.

Proof. We prove that each $n \leq x$, which has a square-free divisor $n_0$ satisfying (i) through (iv) of Lemma 4, has the property stated in the theorem.

Assume that $n$ has such a divisor $n_0 = p_1 p_2 \cdots p_k(n_0, n/n_0) = 1$. Then each Sylow $p_i$-subgroup of $G$, $S_{p_i}(G), 1 \leq i \leq k$, is a normal subgroup (cyclic, of order $p_i$) of $G$, by property (ii) of Lemma 4, applied to the total number $d_i$ of Sylow $p_i$-subgroups of $G$. Moreover, since the image, under any automorphism

2 We thank the referee for simplifying the original proof (by induction) and also showing that the subgroup is characteristic.
of \( G \), of an element of order \( p_i \) is another element of order \( p_i \), each \( S_{p_i}(G) \) is characteristic in \( G \). Also, \( S_{p_i} \cap S_{p_j} \) is the identity subgroup, for each pair \( i \neq j, 1 \leq i, j \leq k \). Thus \( g_i g_j = g_j g_i \) for each such \( i, j \), since \( g_i g_j g_i^{-1} g_j^{-1} \in S_{p_i} \cap S_{p_j} \), by normality. If \( g_i \) generates \( S_{p_i}(G) \), the product \( g_1 g_2 \cdots g_k \) is therefore an element of order \( p_1 p_2 \cdots p_k = n_0 \), and so generates a (cyclic) subgroup \( H \) of order \( n_0 \). Since \( H \) is generated by the characteristic subgroups \( S_{p_i}(G) \), it is also a characteristic subgroup of \( G \).

Remark. Let \( \epsilon_x \) be a (positive) function tending to 0 arbitrarily slowly as \( x \to \infty \). From Lemma 3, with \( g(x) = x^{e_x} \) and \( h(x) = \sqrt{x} \), almost every \( n < x \) has a prime factor \( p > x^{e_x} \); and almost none of these integers has a nontrivial divisor \( \equiv 1 \mod p \), by Lemma 2. Thus for almost all \( n \) every group of order \( n \) has a normal Sylow \( p \)-subgroup of order \( p > n^{\epsilon_x} \).

Corollary. Given \( \epsilon > 0 \), almost all \( n < x \) have the property that \( k(G) > n^{1-\epsilon} \) for each group \( G \) of order \( n \).

Proof. Suppose \( G \) is a group of order \( n \), and \( H \) a cyclic subgroup of \( G \), of order \( n_0 > n^{1-\epsilon/2} \). Let the (complete) conjugacy class (in \( G \)) of \( h \in H \) be denoted by \([h]\), and the centralizer (in \( G \)) of \( h \) by \( C(h) \).

Summing over the \( k_G(H) \) distinct classes (of \( G \)) in \( H \) we have

\[
n_0 = |H| = \sum_{[h]} |H \cap [h]| \leq \max_{h \in H} |[h]| \cdot k_G(H) \\
\leq \frac{n \cdot k(G)}{\min_{h \in H} |C(h)|} \leq \frac{n}{n_0} \cdot k(G),
\]

or \( k(G) \geq \frac{n^2}{n} > n^{1-\epsilon} \).

Remark. Erdös comments that by more complicated number theoretic methods one can prove that as \( f(n) \to \infty \) arbitrarily slowly almost every \( n \) has a square-free divisor \( d > n/(\log n)^{f(n)} \) so that \( (d, n/d) = 1 \) and, for every \( p \mid d, n \) has no nontrivial divisor \( \equiv 1 \mod p \). This is best possible and leads to an improvement of the main theorem, replacing \( n^{1-1/(\log n)^{f(n)}} \) by \( n^{1-(f(n)\log \log n)/\log n} \).

References