QUASI-UNMIXEDNESS AND INTEGRAL CLOSURE OF REES RINGS

PETER G. SAWTELLE

Abstract. For certain Rees rings $\mathfrak{R}$ of a local domain $R$, the quasi-unmixedness of $R$ is characterized in terms of a certain transform of $\mathfrak{R}$ being contained in the integral closure of $\mathfrak{R}$.

1. Introduction. In this paper, a ring shall be a commutative ring with identity. The terminology is basically that of [2] and [12].

Relations between quasi-unmixedness and integral extensions are well known (e.g., [1], [5] and [7]). Also, the study of properties of a ring $R$ via transition to a Rees ring $\mathfrak{R} = \mathfrak{A}(R, A)$ of $R$ (conditions on the ideal $A$ depending on the particular discussion) has often been useful. In particular, characterizations of the quasi-unmixedness of $R$ are given in [10] in terms of localizations of $\mathfrak{R}$ containing $R$ as a quasi-subspace. The $\mathfrak{A}$-algebra $\mathfrak{T} = \mathfrak{T}(u \mathfrak{R})$ (Definition 1) is used in [8] to characterize unmixed local domains. Here, equivalences to the quasi-unmixedness of $R$ are given in terms of $\mathfrak{T}$ being contained in the integral closure of $\mathfrak{R}$ (Theorem 2).

2. Preliminary concepts. Let $B = (b_1, \ldots, b_k)R$ be an ideal in a Noetherian ring $R$. Let $t$ be an indeterminant, and let $u = 1/t$. The Rees ring $\mathfrak{R} = \mathfrak{R}(R, B)$ of $R$ with respect to $B$ is the ring $\mathfrak{R} = R[u, tb_1, \ldots, tb_k]$. $\mathfrak{R}$ is a graded Noetherian subring of $R[u, t]$. If $(R, M)$ is a local ring, then $\mathfrak{R}_M = (M, u, tb_1, \ldots, tb_k)$ is the unique maximal homogeneous ideal of $\mathfrak{R}$. Similar to [12, Theorem 11, p. 157], $\mathfrak{R}_M$ is a graded subring of $K[u, t]$, where $K$ is the total quotient ring of $R$. (Throughout, $S'$ will denote the integral closure of ring $S$.)

For an ideal $B$ in a ring $R$, the integral closure of $B$ in $R$, denoted $B_a$, is the set of all elements in $R$ satisfying an equation of the form $x^n + b_1 x^{n-1} + \cdots + b_n = 0$, where $b_i \in B^i$, $i = 1, \ldots, n$. It is known [4, p. 523] that $B_a$ is an ideal in $R$. In particular, if $B = bR$ is a regular principal ideal, then $(bR)_a = \{r \in R; r/b \in R'\} = bR' \cap R$ [6, Lemma 1].

Definition 1. Let $b$ be a regular nonunit in a ring $R$. Define $\mathfrak{T}(bR) = \{ck/bk; ck \in (b^k R)^{(1)}$, for all large $k\}$, where $(b^k R)^{(1)}$ is the set of elements of $R$ that are in each height one primary component of $b^k R$.

Remark. The following are shown in [11].

(1) $\mathfrak{T}(bR)$ is contained in $R'$ if and only if each height one prime divisor of

Received by the editors July 16, 1973.


Key words and phrases. Local ring, quasi-unmixed, integral closure, Rees ring.

1 This paper contains part of the author's doctoral dissertation written at the University of California at Riverside under the direction of Professor Louis J. Ratliff, Jr.

© American Mathematical Society 1976
bR' contracts to a height one prime (divisor of bR) in R.

(2) $b^n \mathfrak{p}(bR)$ is a finite intersection of height one primary ideals. Also $b^n \mathfrak{p}(bR) \cap R = (b^n R)^{(1)}$.

(3) Define $R^{(1)} = \cap \{R_P; P$ is a height one prime divisor of a principal ideal generated by a nonzero divisor in $R\}$, where $(P)$ denotes the set of regular elements in $R - P$. Then $\mathfrak{p}(bR) = R[1/b] \cap R^{(1)}$.

3. Characterizations of quasi-unmixed local domains. Several preliminary results on completions are given to show that the condition $\mathfrak{p} \subseteq \mathfrak{p}'$ is equivalent to a similar condition for the completion $R^*$ of $R$ (Corollary 1). This is used to give equivalences to the quasi-unmixedness of a local domain (Theorem 2).

**Lemma 1.** Let $B$ be an $M$-primary ideal of a local ring $(R, M)$. Let $\mathfrak{p} = \mathfrak{p}(R, B)$. Let $p$ be a prime ideal of $\mathfrak{p}$, with $u\mathfrak{p} \subseteq p$. Then $(M, u)\mathfrak{p} \subseteq p$, and so all prime ideals containing $u\mathfrak{p}$ lie over $M$.

**Proof.** Since $u$ is in $p$, $B = u\mathfrak{p} \cap R \subseteq p \cap R$. But $B$ is $M$-primary, so $M \subseteq p \cap R$, i.e., $M = p \cap R$. Q.E.D.

**Lemma 2.** Let $\mathfrak{p}$ be as in Lemma 1 and $S = \mathfrak{p}(R, B^*)$. Let $\mathfrak{m}$ (resp., $\mathfrak{m}'$) be the maximal homogeneous ideal of $\mathfrak{p}$ (resp., $S$), and let $\mathfrak{m}'$ (resp., $S^*$) be the completion of $\mathfrak{m}$ (resp., $S$) with respect to the $\mathfrak{m}$ (resp., $\mathfrak{m}'$)-adic topology. Then $\mathfrak{m}' = S^*$ is the completion $R_{\mathfrak{m}}^* = (S_{\mathfrak{m}'})^*$ of $R_{\mathfrak{m}}$ and $S_{\mathfrak{m}'}$.

**Proof.** $R_{\mathfrak{m}}$ is a dense subspace of $S_{\mathfrak{m}}$ [8, Lemma 3.2] and $S^*$ (resp., $S^*$) is the natural completion of $R_{\mathfrak{m}}$ (resp., $S_{\mathfrak{m}}$) [3, Theorem 32, p. 434]. Q.E.D.

**Lemma 3.** Let $R, R^*, B, \mathfrak{p}$ and $S$ be as in Lemma 2. Also, assume that $B$ is generated by a system of parameters. Let $\mathfrak{p} = \mathfrak{p}(wR)$ and $\mathfrak{m}^* = \mathfrak{m}(wS)$. Then $N = (M, u)\mathfrak{p}((M, u)) \cap \mathfrak{p}$ (resp., $N^* = (M^*, u)\mathfrak{p}((M^*, u)) \cap \mathfrak{m}'$) is the only prime divisor of $\mathfrak{p}^*$ (resp., $\mathfrak{m}^*$).

**Proof.** By [8, Remark 3.10(ii)], $(M, u)\mathfrak{p}$ is the only height one prime divisor of $u\mathfrak{p}$. By the one-to-one correspondence (and denseness) in [8, Lemma 3.2], $(M^*, u)\mathfrak{m}^* = (M, u)\mathfrak{m}^*$ is the only height one prime divisor of $u\mathfrak{m}^*$, and by the one-to-one correspondence in [11, Lemma 2(9)], $N$ (resp., $N^*$) is the only height one prime divisor of $w\mathfrak{p}$ (resp., $w\mathfrak{m}^*$). By Remark (2), this ideal has no imbedded prime divisors. Q.E.D.

**Theorem 1.** With the notation of Lemma 2, let $p \subseteq P$ be an inclusion of prime ideals in $\mathfrak{p}$ with $u \in p$. Then the following statements hold:

(1) $\mathfrak{p}/p$ is a locally unmixed, pseudo-geometric domain [2, p. 131].

(2) $p\mathfrak{p}^*$ is a semiprime, unmixed ideal in the completion $R^*_p$ of $R_p$.

(3) In the completion $R^*_p$ of $R_p$, $p\mathfrak{p}^*$ has pure height equal to height $p$ and has pure depth equal to depth $p\mathfrak{p}^*_p$.

(4) $p\mathfrak{p}^* = pS^*$ has pure height equal to height $p$, where $p$ is contained in the maximal homogeneous ideal of $\mathfrak{p}$.

**Proof.** Since $p \subseteq R = (M, u)(R/M)(u^*, tB)^u$, where $X^u$ denotes $X$ modulo $p$. Thus $\mathfrak{p}/p$ is finitely generated as a ring over the field
QUASI-UNMIXEDNESS AND INTEGRAL CLOSURE OF REES RINGS

$R/M$, and so is locally unmixed [2, (34.9)], and pseudo-geometric [2, (36.5)].

This shows (1). By localizing to $R_p$, (2) follows from [2, (36.4)] and (1).

For (3), since $pR_p$ is an unmixed ideal (by (2)), it has pure depth equal to depth $pR_p = \text{depth} pR_p$. Since $pR_p$ is semiprime, that it has pure height equal to height $p$ follows from [2, (22.9)]. (4) is a special case of (3) since $\gamma^* = \gamma = S^*$ by Lemma 2. Q.E.D.

**Corollary 1.** Let the notation be as in Lemma 2. Then $\gamma(u\gamma) \subseteq R'$ if and only if $\gamma(uS) \subseteq S'$.

**Proof.** Since $(u^n\gamma)_a = u^n\gamma' \cap \gamma$ and $\gamma'$ and $\gamma$ are graded subrings of $K[u, t]$, it follows that $(u^n\gamma)_a$ is a homogeneous ideal in $\gamma$. Therefore, every prime divisor of $(u^n\gamma)_a$, for $n \geq 1$, and every prime divisor of the homogeneous ideal $u\gamma$ is contained in the maximal homogeneous ideal $\mathfrak{m}$ of $\gamma$. By [11, Lemma 4(2)], $\gamma(u\gamma_{\mathfrak{m}}) \subseteq \gamma_{\mathfrak{m}}$ if and only if $\gamma(u\gamma) \subseteq \gamma'$. Now, let $P$ be a height one prime divisor of $u\gamma_{\mathfrak{m}}$, and $p = P \cap \gamma$. Then $P\gamma_{\mathfrak{m}} = P\gamma^*$ has pure height one (Theorem 1(4)). Therefore, by [11, Corollary 2], $\gamma(u\gamma_{\mathfrak{m}}) \subseteq \gamma_{\mathfrak{m}}$ if and only if $\gamma(u\gamma_{\mathfrak{m}}) \subseteq \gamma_{\mathfrak{m}}^*$. But $\gamma_{\mathfrak{m}}^* = (\gamma_{\mathfrak{m}})^*$ so the last inclusion is equivalent to $\gamma(u\gamma_{\mathfrak{m}})^* \subseteq (\gamma_{\mathfrak{m}})^*$. As above, this is equivalent to $\gamma(u\gamma_{\mathfrak{m}}) \subseteq (\gamma_{\mathfrak{m}}^*)'$, which, again as above, is equivalent to $\gamma(u\gamma) \subseteq S'$. Q.E.D.

**Lemma 4.** Let $b$ be a regular nonunit in a Noetherian ring $R$ and $q$ a minimal prime divisor of zero in $R'$. Then there exists a height one prime divisor $P$ of $bR'$ that contains $q$.

**Proof.** In $R'$, let $Z = \text{rad} (0) = \bigcap_{i=1}^n q_i (q_1 = q)$. Since $Z \subseteq bR'$ [9, Lemma 2.4], we may pass to $R'/Z = \overline{R}$. $\overline{R}$ is the direct sum of Krull domains $\bigoplus_{i=1}^n R'/q_i = \bigoplus_{i=1}^n \overline{R}e_i$, where the $e_i$ are the associated orthogonal idempotents. A height one prime divisor $p_1$ of $be_1$ in $\overline{R}e_1$ gives rise to the desired $P$. Q.E.D.

**Theorem 2** (cf. [8, Theorem 5.17]). Let $(R, M)$ be a local domain of altitude $n \geq 1$. Then the following statements are equivalent:

1. $R$ is quasi-unmixed.

2. For every finitely generated domain $A$ over $R$, and for each multiplicatively closed subset $S$ of $A$, $(A_S)^{(1)} \subseteq A_S'$.

3. For every ideal $B$ in $R$, $\gamma(u\gamma) \subseteq \gamma'$, where $\gamma = \gamma(R, B)$.

4. There exists an $M$-primary ideal $B$ in $R$ that is generated by a system of parameters such that $\gamma(u\gamma) \subseteq \gamma'$, where $\gamma = \gamma(R, B)$.

**Proof.** (1 $\Rightarrow$ 2). By [11, Lemma 1(3) and (5)], it is sufficient to show $A^{(1)} \subseteq A'$. By [5, Corollary 2.5], $A$ is locally quasi-unmixed. Then, by [7, Theorem 3.8], each height one prime ideal in $A'$ contracts to a height one prime in $A$. Thus, by [8, Corollary 5.7], $A^{(1)} \subseteq A'$.

(2 $\Rightarrow$ 3). Since $\gamma$ is a finite extension of $R$, $\gamma^{(1)} \subseteq \gamma'$, by hypothesis. And, $\gamma(u\gamma) \subseteq \gamma^{(1)}$.

(3 $\Rightarrow$ 4) is obvious.

(4 $\Rightarrow$ 1). Let $B$ be an $M$-primary ideal of $R$ generated by a system of parameters. Let $\gamma = \gamma'$, where $\gamma = \gamma(R, B)$ and $\gamma = \gamma(u\gamma)$. By Corollary 1, $\gamma^* \subseteq S'$, where $S = \gamma(R^*, BR^*)$ and $\gamma^* = \gamma(uS)$ ($R^*$ is the completion of...
R). Let $q$ be a minimal prime divisor of zero in $S$. Let $q'$ be the minimal prime divisor of zero in $S'$ that lies over $q$ ($S$ and $S'$ have the same total quotient ring). By Lemma 4, there exists a height one prime divisor $p'$ of $uS'$ that contains $q'$. By Remark 1, $p' \cap S = p$ is a height one prime divisor of $uS$. Hence, $q \subseteq p = (M^*, u)S$ (Lemma 3). Since $q$ was an arbitrary minimal prime divisor of zero in $S$, $R$ is quasi-unmixed [10, Corollary 9]. Q.E.D.

By combining Theorem 2 and the Remark, further characterizations of the quasi-unmixedness of $R$ can be obtained.

**Bibliography**


Department of Mathematics, University of Missouri, Rolla, Missouri 65401