ON A CONJECTURE OF S. CHOWLA

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ABSTRACT. Let $\psi(x) = x - [x] - \frac{1}{2}$. It has been conjectured by S. Chowla that $\sum_{n \leqslant \sqrt{x}} \{\psi^2(x/n) - 1/12\} = O(x^{1/4+\epsilon})$, for every $\epsilon > 0$. In this paper we show that this conjecture is equivalent to $\sum_{n \leqslant \sqrt{x}} n^2 \{\psi^2(x/n) - 1/12\} = O(x^{5/4+\epsilon})$ by proving that

$$\sum_{n \leqslant \sqrt{x}} \left\{\psi^2\left(\frac{x}{n}\right) - \frac{1}{12}\right\} + \frac{1}{x} \sum_{n \leqslant \sqrt{x}} n^2 \left\{\psi^2\left(\frac{x}{n}\right) - \frac{1}{12}\right\} = O(x^{1/4}).$$

1. **Introduction.** Throughout the following, x denotes a real variable $\geqslant 1$, $[x]$ denotes the greatest integer $\leqslant x$ (integral part of x) and $\psi(x) = x - [x] - \frac{1}{2}$.

Let $\tau(n)$ denote the number of divisors of a positive integer n and let

$$(1.1) \qquad \Delta(x) = \sum_{n \leqslant x} \tau(n) - \{x \log x + (2\gamma - 1)x\},$$

where γ is Euler's constant.

The classical 'Dirichlet divisor problem' states that

$$(1.2) \qquad \Delta(x) = O(x^{1/4+\epsilon}) \quad \text{for every } \epsilon > 0.$$

This problem is still open. The best result known till now is due to G. A. Kolesnik [3], who proved that

$$(1.3) \qquad \Delta(x) = O(x^{346/1067+\epsilon}) \quad \text{for every } \epsilon > 0.$$

It can be seen that (1.2) is equivalent to the conjecture (see, for example, [1, equation (10)]) that

$$(1.4) \qquad \sum_{n \leqslant \sqrt{x}} \psi\left(\frac{x}{n}\right) = O(x^{1/4+\epsilon}) \quad \text{for every } \epsilon > 0.$$

In 1963 S. Chowla and H. Walum [1] proved the following result, which is analogous to (1.4):

$$(1.5) \qquad \sum_{n \leqslant \sqrt{x}} n \left\{\psi^2\left(\frac{x}{n}\right) - \frac{1}{12}\right\} = O(x^{3/4}).$$

In 1965, S. Chowla (cf. [2, pp. 90-91]) states "It is also likely that

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(1.6)
$$\sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$= O(x^{1/4+\epsilon}) \quad \text{for every } \epsilon > 0.$$

Although (1.6) resembles (1.5), it is probably extremely hard to prove (1.6) like (1.4). The difference is, of course, that (somewhat surprisingly), (1.5) is not hard to prove".

In this paper we prove the following

THEOREM. *We have*

(1.7)
$$\sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$+ \frac{1}{x} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} = O(x^{1/4}).$$

From this theorem, we immediately have the following

COROLLARY. *The conjecture of S. Chowla, namely (1.6), is equivalent to*

(1.8)
$$\sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$= O(x^{5/4+\epsilon}) \quad \text{for every } \epsilon > 0.$$

REMARK. It follows easily from (1.5) and the Theorem that

(1.9)
$$\sum_{n \leqslant \sqrt{x}} (\sqrt{x} - n)^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} = O(x^{5/4}),$$

and

(1.10)
$$\sum_{n \leqslant \sqrt{x}} (\sqrt{x} + n)^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} = O(x^{5/4}).$$

It is highly desirable to have direct proofs of (1.9) and (1.10) without appealing to (1.5) and the Theorem. It can be easily seen that any two of (1.5), (1.7), (1.9) and (1.10) imply the other two.

2. **Prerequisites.** We need the following two lemmas due to S. L. Segal [4]:

LEMMA 1. *(Cf. [4, Lemma, p. 279 and p. 765].) Let f(n) be a function of a positive integral variable, and suppose*

$$\sum_{n \leqslant x} f(n) = g(x) + E(x),$$

where g(x) is twice continuously differentiable, and g″(x) is of constant sign, for x ⩾ 1. Then

$$\sum_{n \leqslant x} E(n) = \tfrac{1}{2}g(x) + (\tfrac{1}{2} - \psi(x))E(x)$$

$$+ \int_1^x E(t)\,dt + O(|g'(x)|) + O(1).$$

LEMMA 2. (*Cf.* [4, *Theorem 1, p. 281*].) *Let* $\sigma(n)$ *denote the sum of all the divisors of* n, *and*

(2.1)
$$E_1(x) = \sum_{n \leqslant x} \sigma(n) - \frac{\pi^2 x^2}{12}.$$

Then we have

(2.2)
$$\sum_{n \leqslant x} E_1(n) = \left(\frac{\pi^2}{24} - \frac{1}{4}\right)x^2 + O(x^{5/4}).$$

3. **Proof of the Theorem.** Let $S(x) = \sum_{n \leqslant x} \sigma(n)$. Then we have by (2.1),

(3.1)
$$S(x) = \pi^2 x^2/12 + E_1(x).$$

Let

(3.2)
$$E_1'(x) = S'(x) - \pi^2 x^3/18,$$

where

(3.3)
$$S'(x) = \sum_{n \leqslant x} \sigma(n)n.$$

We have

$$S(x) = \sum_{n \leqslant x} \sigma(n) = \sum_{d\delta \leqslant x} d = \sum_{d\delta \leqslant x; d \leqslant \sqrt{x}} d + \sum_{d\delta \leqslant x; \delta \leqslant \sqrt{x}} d - \sum_{d \leqslant \sqrt{x}; \delta \leqslant \sqrt{x}} d$$

$$= \sum_{d \leqslant \sqrt{x}} d\left[\frac{x}{d}\right] + \frac{1}{2}\sum_{\delta \leqslant \sqrt{x}} \left\{\left[\frac{x}{\delta}\right]^2 + \left[\frac{x}{\delta}\right]\right\} - \tfrac{1}{2}[\sqrt{x}]\{[\sqrt{x}]^2 + [\sqrt{x}]\}.$$

Since

(3.4)
$$[\sqrt{x}] = \sqrt{x} - \psi(\sqrt{x}) - \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

and

(3.5)
$$\sum_{n > \sqrt{x}} \frac{1}{n^2} = \frac{1}{[\sqrt{x}]} - \frac{1}{2[\sqrt{x}]^2} + \frac{1}{6[\sqrt{x}]^3} + O\left(\frac{1}{x^2}\right),$$

we have

$$S(x) = \sum_{n \le \sqrt{x}} n \left\{ \frac{x}{n} - \psi\left(\frac{x}{n}\right) - \frac{1}{2} \right\}$$

$$+ \frac{1}{2} \sum_{n \le \sqrt{x}} \left\{ \left(\frac{x}{n} - \psi\left(\frac{x}{n}\right) - \frac{1}{2}\right)^2 + \left(\frac{x}{n} - \psi\left(\frac{x}{n}\right) - \frac{1}{2}\right) \right\}$$

$$- \tfrac{1}{2}(\sqrt{x} - \psi(\sqrt{x}) - \tfrac{1}{2})\{x + \psi^2(\sqrt{x}) - 2\sqrt{x}\psi(\sqrt{x}) - \tfrac{1}{4}\}$$

$$= x\left(\sqrt{x} - \psi(\sqrt{x}) - \frac{1}{2}\right) - \sum_{n \le \sqrt{x}} n\psi\left(\frac{x}{n}\right) - \frac{1}{4}(x - 2\sqrt{x}\psi(\sqrt{x}) + O(1))$$

$$+ \frac{x^2}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{x^2}{2} \sum_{n > \sqrt{x}} \frac{1}{n^2} + \frac{1}{2} \sum_{n \le \sqrt{x}} \psi^2\left(\frac{x}{n}\right) - x \sum_{n \le \sqrt{x}} \frac{1}{n}\psi\left(\frac{x}{n}\right)$$

$$- \tfrac{1}{8}\sqrt{x} - \tfrac{1}{2}x^{3/2} - \tfrac{3}{2}\sqrt{x}\psi^2(\sqrt{x}) + \tfrac{3}{2}x\psi(\sqrt{x}) + \tfrac{1}{8}\sqrt{x}$$

$$+ x/4 - \tfrac{1}{2}\sqrt{x}\psi(\sqrt{x}) + O(1)$$

$$= \frac{\pi^2 x^2}{12} + \frac{1}{2}x^{3/2} - \sum_{n \le \sqrt{x}} n\psi\left(\frac{x}{n}\right) - x \sum_{n \le \sqrt{x}} \frac{1}{n}\psi\left(\frac{x}{n}\right)$$

$$+ \frac{1}{2} \sum_{n \le \sqrt{x}} \psi^2\left(\frac{x}{n}\right) - \frac{x}{2} - \frac{1}{2}x^{3/2}\left(1 - \frac{\psi(\sqrt{x})}{\sqrt{x}} - \frac{1}{2\sqrt{x}}\right)^{-1}$$

$$+ \frac{x}{4}\left(1 - \frac{\psi(\sqrt{x})}{\sqrt{x}} - \frac{1}{2\sqrt{x}}\right)^{-2} - \frac{1}{12}\sqrt{x}\left(1 - \frac{\psi(\sqrt{x})}{\sqrt{x}} - \frac{1}{2\sqrt{x}}\right)^{-3}$$

$$- \tfrac{3}{2}\sqrt{x}\psi^2(\sqrt{x}) + \tfrac{1}{2}x\psi(\sqrt{x}) + O(1)$$

$$= \frac{\pi^2 x^2}{12} - \sum_{n \le \sqrt{x}} n\psi\left(\frac{x}{n}\right) - x \sum_{n \le \sqrt{x}} \frac{1}{n}\psi\left(\frac{x}{n}\right)$$

$$+ \frac{1}{2} \sum_{n \le \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$- \frac{x}{2} + \frac{1}{12}\sqrt{x} - 2\sqrt{x}\psi^2(\sqrt{x}) + O(1),$$

after simplification. Hence by (3.1), we have

$$(3.6) \quad E_1(x) = - \sum_{n \le \sqrt{x}} n\psi\left(\frac{x}{n}\right) - x \sum_{n \le \sqrt{x}} \frac{1}{n}\psi\left(\frac{x}{n}\right)$$

$$+ \frac{1}{2} \sum_{n \le \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$- x/2 + \tfrac{1}{12}\sqrt{x} - 2\sqrt{x}\psi^2(\sqrt{x}) + O(1).$$

Also, we have

$$S'(x) = \sum_{n \leqslant x} \sigma(n)n = \sum_{d\delta \leqslant x} d^2 \delta$$

$$= \sum_{d\delta \leqslant x; d \leqslant \sqrt{x}} d^2\delta + \sum_{d\delta \leqslant x; \delta \leqslant \sqrt{x}} d^2\delta - \sum_{d \leqslant \sqrt{x}; \delta \leqslant \sqrt{x}} d^2\delta$$

$$= \frac{1}{2} \sum_{d \leqslant \sqrt{x}} d^2 \left\{ \left[\frac{x}{d}\right]^2 + \left[\frac{x}{d}\right] \right\}$$

$$+ \frac{1}{6} \sum_{\delta \leqslant \sqrt{x}} \delta \left\{ \left(\left[\frac{x}{\delta}\right]^2 + \left[\frac{x}{\delta}\right] \right) \left(2\left[\frac{x}{\delta}\right] + 1 \right) \right\}$$

$$- \frac{1}{12} ([\sqrt{x}]^2 + [\sqrt{x}])^2 (2[\sqrt{x}] + 1)$$

$$= \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \frac{x^2}{n^2} + \psi^2\left(\frac{x}{n}\right) - 2\frac{x}{n}\psi\left(\frac{x}{n}\right) - \frac{1}{4} \right\}$$

$$+ \frac{1}{3} \sum_{n \leqslant \sqrt{x}} n \left\{ \left(\frac{x^2}{n^2} + \psi^2\left(\frac{x}{n}\right) - 2\frac{x}{n}\psi\left(\frac{x}{n}\right) - \frac{1}{4} \right) \left(\frac{x}{n} - \psi\left(\frac{x}{n}\right) \right) \right\}$$

$$- \frac{1}{6} (x + \psi^2(\sqrt{x}) - 2\sqrt{x}\psi(\sqrt{x}) - \frac{1}{4})^2 (x - \psi(\sqrt{x})).$$

Now making use of (3.4) and (3.5), we get after simplification:

$$S'(x) = \frac{\pi^2 x^3}{18} - x \sum_{n \leqslant \sqrt{x}} n\psi\left(\frac{x}{n}\right) - x^2 \sum_{n \leqslant \sqrt{x}} \frac{1}{n}\psi\left(\frac{x}{n}\right)$$

$$+ x \sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} + \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$- x^2/4 + \frac{1}{12} x^{3/2} - 2x^{3/2}\psi^2(\sqrt{x}) + O(x).$$

Hence by (3.2), we have

$$E_1'(x) = -x \sum_{n \leqslant \sqrt{x}} n\psi\left(\frac{x}{n}\right) - x^2 \sum_{n \leqslant \sqrt{x}} \frac{1}{n}\psi\left(\frac{x}{n}\right)$$

(3.7)
$$+ x \sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} + \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

$$- x^2/4 + \frac{1}{12} x^{3/2} - 2x^{3/2}\psi^2(\sqrt{x}) + O(x).$$

Now, by (3.6) and (3.7), we have

$$E_1'(x) - xE_1(x) = \frac{x}{2} \sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$

(3.8)
$$+ \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} + \frac{x^2}{4} + O(x).$$

By partial summation, (3.1) and (3.3), we have

$$S'(x) = \sum_{n \leqslant x} \sigma(n)n = xS(x) - \int_1^x S(t)\,dt$$

$$= xS(x) - \int_1^x \left\{ \frac{\pi^2 t^2}{12} + E_1(t) \right\} dt,$$

so that

(3.9) $$\qquad S'(x) - xS(x) = -\frac{\pi^2}{36}(x^3 - 1) - \int_1^x E_1(t)\,dt.$$

But by (3.1) and (3.2),

(3.10) $$\qquad S'(x) - xS(x) = \frac{\pi^2 x^3}{18} - \frac{\pi^2 x^3}{12} + E_1'(x) - xE_1(x).$$

Hence by (3.9) and (3.10), we have

(3.11) $$\qquad \int_1^x E_1(t)\,dt = -(E_1'(x) - xE_1(x)) + \pi^2/36.$$

Hence by (3.8) and (3.11),

(3.12)
$$\int_1^x E_1(t)\,dt = -\frac{x}{2} \sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$
$$- \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} - \frac{x^2}{4} + O(x).$$

Now, taking $f(n) = \sigma(n)$, $g(x) = \pi^2 x^2/12$, $E(x) = E_1(x)$, we see that the hypothesis of Lemma 1 is satisfied, by virtue of (3.1). Hence by Lemma 1, we have

(3.13)
$$\sum_{n \leqslant x} E_1(n) = \frac{\pi^2 x^2}{24} + (\tfrac{1}{2} - \psi(x))E_1(x) + \int_1^x E_1(t)\,dt$$
$$+ O(x) + O(1)$$
$$= \frac{\pi^2 x^2}{24} + \int_1^x E_1(t)\,dt + O(x \log x),$$

since it is well known that $E_1(x) = O(x \log x)$. Hence by (3.12) and (3.13), we have

(3.14)
$$\sum_{n \leqslant x} E_1(n) = \left(\frac{\pi^2}{24} - \frac{1}{4} \right)x^2 - \frac{x}{2} \sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\}$$
$$- \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} + O(x \log x),$$

so that by Lemma 2, it follows that

$$\frac{x}{2} \sum_{n \leqslant \sqrt{x}} \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} + \frac{1}{2} \sum_{n \leqslant \sqrt{x}} n^2 \left\{ \psi^2\left(\frac{x}{n}\right) - \frac{1}{12} \right\} = O(x^{5/4}).$$

Hence the Theorem follows.

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