ON THE NONEXISTENCE OF GROUPS WITH EXTRA-SPECIAL COMMUTATOR SUBGROUP

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ABSTRACT. In this paper, we extend a result of Joseph and Finkelstein and show that there is no group $G$ such that $G'$ is an extra-special $p$-group of exponent $\geq p$ ($p$ odd).

K. Joseph and L. Finkelstein [2] have shown that if $p$ is an odd prime, there does not exist a finite group $G$ satisfying the following three conditions:

(i) $G'$ is an extra-special $p$-group of exponent $\geq p$.
(ii) $Z(G) \subseteq G'$.
(iii) $G$ acts irreducibly on $G'/Z(G')$.

It is the object of this paper to prove that their result remains valid even if conditions (ii) and (iii) are dropped. That is, there is no finite group $G$ such that $G'$ is an extra-special $p$-group of exponent $\geq p$ ($p$ odd).

Recall that a finite $p$-group $G$ is called extra-special if $Z(G) = G^p$, and $|G'| = p$. We now list a series of lemmas which will be needed for the main theorem. In all that follows, $p$ is an odd prime.

**Lemma 1.** Let $G$ be an extra-special $p$-group. Then

(a) $(xy)^p = x^p y^p$,

(b) $x^p \in Z(G)$ for all $x, y \in G$.

**Proof.** See [1, p. 183].

**Lemma 2.** Let $G$ be an extra-special $p$-group of exponent $\geq p$, and let $U = \{x \in G | x^p = 1\}$. Then $U$ is a characteristic subgroup of $G$ and $[G : U] = p$.

**Proof.** The fact that $U$ is a characteristic subgroup follows immediately from Lemma 1(a), and the fact that automorphisms preserve order. The map $x \to x^p$ is a homomorphism of $G$ onto $Z(G)$ with kernel $U$. Hence $G/U \cong Z(G)$ and $[G : U] = p$.

**Lemma 3.** Let $G$ be a finite $p$-group of linear transformations acting on a vector space $V$ over a field $F$ of characteristic $p$. Then some nonzero vector of $V$ is fixed by every element of $G$.

**Proof.** See [1, p. 31].

**Lemma 4.** Suppose an abelian group $G$ acts as a group of linear transformations
on a vector space $V$ over a field $F$. Let $S$ be a subspace of $V$, and $H$ a subgroup of $G$ whose elements induce scalar transformations on $S$. Let $S^G$ be the subspace of $V$ generated by all vectors $s^g$, $s \in S$, $g \in G$. Then $H$ also induces scalar transformations on $S^G$.

**Proof.** Let $s_1^g$, $s_2^g \in S^G$, and suppose $h \in H$ with $s^h = \lambda s$ for all $s \in S$. Then

$$(s_1^g + s_2^g)^h = s_1^{g^h} + s_2^{g^h} = (s_1^h)^g + (s_2^h)^g = (\lambda s_1)^g + (\lambda s_2)^g.$$

Similarly, if $c \in F$, then

$$(cs_1^g)^h = c(s_1^g)^h = c(s_1^h)^g = \lambda (cs_1^g).$$

The lemma follows.

**Theorem.** Let $G$ be an extra-special $p$-group ($p > 2$) of exponent $> p$. Then there is no finite group $K$ such that $K' = G$.

**Proof.** Suppose $K' = G$. Then $K$ acts by conjugation on $G'$. Moreover, $K$ acts in a natural way on $G/G' = \overline{G}$; namely, if $k \in K$, and $aG' \in G'$, then $(aG')^k = (k^{-1}ak)G'$. This is easily seen to be well defined. Now $\overline{K} = K/G$ is abelian, so we can write $\overline{K} = \overline{K}_p \times \overline{K}_p$, where $\overline{K}_p$ is a $p$-group, and $\overline{K}_p$ has order prime to $p$. The group $\overline{K}$ acts in a natural way on $\overline{G}$. Indeed, if $K = K/G$ is $\overline{K}$-invariant, then we define $(aG')^x = (k^{-1}ak)G'$. To see that this is well defined, suppose $x = IG$ and $aG' = bG'$. We need to show that $(k^{-1}ak)G' = (l^{-1}bl)G'$, or equivalently that $l^{-1}b^{-1}k^{-1}ak \in G'$. But $l^{-1}b^{-1}k^{-1}ak = l^{-1}b^{-1}k^{-1}[a^{-1}]^t(b^{-1})bl \in G'$, since $kl^{-1} \in G$, and $ab^{-1} \in G'$.

Now $\overline{G}$ is elementary abelian, so it is a vector space over $\mathbb{Z}_p$. Let $U = \{x \in G | x^p = 1\}$, and define $\overline{U} = U/G'$. By Lemma 2, $U$ has index $p$ in $G$, and $\overline{U}$ is a subspace of $\overline{G}$. As $U$ is characteristic in $G$, $\overline{U}$ is $\overline{K}_p$-invariant, so by Maschke's Theorem [1, p. 66], there exists a $\overline{K}_p$-invariant subspace $W \subseteq \overline{G}$ such that $\overline{G} = \overline{U} \oplus W$. Let $\overline{W} = W/G'$. As $[G:U] = p$, $W$ must have order $p^2$. Furthermore, $W$ is cyclic since $W \not\subseteq U$.

The action of $\overline{K}$ on $\overline{G}$ is given by a character $\lambda$: $\overline{K} \rightarrow \mathbb{Z}$; that is, if $k \in \overline{K}$, then $c^k = c^{\lambda(k)}$ for all $c \in \overline{G}$. Clearly then, $\overline{K}_p$ acts with character $\lambda$ on $\overline{G}$, hence also on $\overline{W}$. Moreover, $\overline{W}$ is not $K$-invariant. For suppose it were. Then $\overline{W}$ would be normal in $K$, and $N_K(\overline{W})/C_K(\overline{W}) = K/C_K(\overline{W})$ would be abelian, which implies that $\overline{W} \subseteq Z(G)$, a contradiction. Thus $\overline{W}^\overline{K} \cap \overline{U} \neq \{1\}$, where $\overline{W}^\overline{K} = \langle w^k | w \in \overline{W}, k \in \overline{K} \rangle$. The $p$-group $\overline{K}_p$ acts on $\overline{W}^\overline{K} \cap \overline{U}$, so by Lemma 3, there is a subgroup $V \subseteq \overline{W}^\overline{K} \cap \overline{U}$, of order $p$, which is elementwise fixed by $\overline{K}_p$.

As $\overline{K}_p$ acts with character $\lambda$ on $\overline{W}$, by Lemma 4, it acts with character $\lambda$ on $\overline{W}^\overline{K}$, in particular on $\overline{V}$. If $\overline{V} = \overline{V}/G'$, then $\overline{V}$ is elementary abelian of order $p^2$, and since $\overline{V}$ is $\overline{K}$-invariant, $V \subseteq K$.

Let $K_p$ and $K_{p'}$ be defined by $K_p/G = \overline{K}_p$ and $K_{p'}/G = \overline{K}_p$. Since $\overline{K}_p$ fixes
$V$ elementwise, and $K_p$ fixes $G'$ elementwise, $K_p$ must act on $V$ via matrices of the form $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$. Also, the above discussion shows that $K_p'$ acts on $V$ via matrices of the form $(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix})$.

We conclude that $K$ acts on $V$ via an abelian group of matrices, since matrices of the form $(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})$ commute. As $K' = G$, $G$ acts trivially on $V$, that is $V \subseteq Z(G)$. This is a contradiction, and the theorem follows.

It might be worthwhile to note that if a group $G$ is the commutator of any group $K$, then $G$ is also the commutator of a finite group [3]. Hence the theorem is true without the restriction that $K$ be finite.

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REFERENCES

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