WEAK CONVERGENCE OF SEMIGROUPS IMPLIES STRONG CONVERGENCE OF WEIGHTED AVERAGES

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Abstract. For a fixed \( p, 1 < p < \infty \), let \( \{\mathcal{T}_t : t > 0\} \) be a strongly continuous semigroup of positive contractions on \( L_p \) of a \( \sigma \)-finite measure space. We show that weak convergence of \( \{\mathcal{T}_t : t > 0\} \) in \( L_p \) is equivalent with the strong convergence of the weighted averages \( \int_0^\infty \mathcal{T}_t f \mu_n(dt) \) (\( n \to \infty \)) for every \( f \in L_p \) and every sequence \( (\mu_n) \) of signed measures on \((0, \infty)\), satisfying \( \sup_n \|\mu_n\| < \infty \); \( \lim_n \mu_n(0, \infty) = 1 \); and for each \( d > 0 \), \( \limsup_{n \to \infty} \sup_{t > 0} |\mu_n(t, c + d)| = 0 \). The positivity assumption is not needed if \( p = 1 \) or \( 2 \). We show that such a result can be deduced—not only in \( L_p \) but in general Banach spaces—from the corresponding discrete parameter version of the theorem.

In recent years, various authors have studied the relations between weak and strong operator convergence: Blum-Hanson [4], Hanson-Pledger [7], Lin [9], Akcoglu-Sucheston [1], Jones-Kuftinec [8], Fong-Sucheston [6], and very recently Akcoglu-Sucheston [2], [3], who proved the theorem for positive \( L_p \)-contractions, \( 1 < p < \infty \). This theorem [1], [6], [3] states that if \( T \) is a positive contraction on \( L_p \) of a \( \sigma \)-finite measure space \((X, \mathcal{E}, m)\), where \( p \) is fixed and \( 1 < p < \infty \), then weak-\( \lim_{n \to \infty} T^n f \) (w-\( \lim_{n} T^n f \)) exists for each \( f \in L_p \) if, and only if, \( \lim_{n \to \infty} \sum_{m=1}^{\infty} a_{nm} T^m f \) exists for every \( f \in L_p \) and every matrix \( (a_{nm}) \) with real entries satisfying

\[
(1.1) \quad \sup_n \sum_m |a_{nm}| < \infty; \lim_n \sum_m a_{nm} = 1; \lim \max_n |a_{nm}| = 0.
\]

It has also been shown that the positivity assumption is not needed if \( p = 1 \) or \( 2 \) [1], [6]. The problem of whether or not positivity is needed for \( p \neq 1 \) or \( 2 \) is still open. Matrices satisfying \((1.1)\) were introduced in ergodic context in [6] and have been called uniformly regular; we denote the class of all uniformly regular matrices by \( \mathfrak{M}_R \). Intuitively, a matrix \( (a_{nm}) \in \mathfrak{M}_R \) if and only if it is properly averaging, in the sense that the masses \( a_{nm} \) spread as \( n \to \infty \).

A semigroup \( \{\mathcal{T}_t : t > 0\}, \mathcal{T}_t T_s = T_{t+s} \), of linear operators on a Banach space \( B \) is called strongly continuous if for each \( x \in B \) and each \( s > 0 \), \( \lim_{t \to s} \|\mathcal{T}_t x - T_s x\| = 0 \). R. Sato [10] recently obtained the following continuous parameter version of the strong ergodic theorem: For a fixed function \( f \) in \( L_2 \) and a strongly continuous semigroup \( \{\mathcal{T}_t : t > 0\} \) of contractions on \( L_2 \), \( \text{w-\( \lim_{t \to \infty} T_t f = f_0 \) implies \( \lim_{n \to \infty} \int_0^\infty a_n(t) T_t f dt = f_0 \) for every sequence \( (a_n) \) of nonnegative, Lebesgue integrable functions on \((0, \infty)\) satisfying
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\[ J_0^\infty a_n(t) = 1 \text{ and } \lim_{n \to \infty} \|a_n\|_\infty = 0. \] In this note, we show that a stronger result can be deduced—not only in \( L_2 \), but in general Banach spaces—from the corresponding discrete version of the theorem (§2). In §3, we obtain as corollaries the continuous parameter version of the Akcoglu-Sucheston theorem for the Banach spaces \( L_p(X, \mathcal{E}, m), \ 1 \leq p < \infty \).

2. A linear operator \( T \) on a Banach space \( B \) is called \textit{power-bounded} if \( \sup_n \|T^n\| < \infty \); a semigroup \( \{T_t: t > 0\} \) of linear operators on \( B \) is called \textit{uniformly bounded} if \( \sup_{t > 0} \|T_t\| < \infty \). The \textit{total variation} of a signed measure \( \mu \) is denoted by \( |\mu| \). We denote by \( \mathfrak{A} \) the family of all sequences \( (\mu_n) \) of signed measures on the \( \sigma \)-algebra of Lebesgue measurable subsets of \( (0, \infty) \) satisfying

\[
\text{(2.1) } \sup_n \|\mu_n\| < \infty; \quad \lim_{n \to \infty} \mu_n(0, \infty) = 1;
\]

(2.1)

\[
\text{Remark 1. If } (a_n) \text{ is a sequence of Lebesgue integrable functions on } (0, \infty) \text{ satisfying}
\]

\[
\text{sup}_n \int_0^\infty |a_n(t)| \, dt < \infty; \quad \lim_{n \to \infty} \int_0^\infty a_n(t) \, dt = 1;
\]

(2.2)

\[
\text{lim}_{n \to \infty} \sup_{c > 0} \int_c^{c+d} |a_n(t)| \, dt = 0
\]

for each \( d > 0 \), and if we set \( d\mu_n = a_n \, dt \), then \( (\mu_n) \in \mathfrak{A} \). We note that sequences \( (a_n) \) satisfying (2.2) include those considered by Sato in [10].

\text{Remark 2. Let } \delta(t) \text{ denote the unit point mass at } t. \text{ If } t_m > 0, t_m \to \infty, \quad (a_{nm}) \in \mathfrak{A}, \text{ and if we set } \mu_n = \sum_m a_{nm} \delta(t_m), \text{ then } (\mu_n) \in \mathfrak{A}.

\text{Remark 3. If } x(t) \text{ is a bounded continuous function from } (0, \infty) \text{ to a Banach and if we set } d\mu_n = a_n \, dt, \text{ then } (\mu_n) \in \mathfrak{A}. \text{ We note that sequences } (a_n) \text{ satisfying (2.2) include those considered by Sato in [10].}

\[
\left\| \int_0^\infty x(t)\mu(dt) \right\| \leq \left( \sup_{i \geq 0} \|x(t)\| \right) \cdot \|\mu\| \quad (\text{cf. [5]}).
\]

\text{Theorem 2.1. Let } x \text{ be a fixed element in a Banach space } B, \text{ real or complex. Then } (\alpha) \text{ implies } (\beta): \]

(\alpha) For every power bounded linear operator \( T \) on \( B \), if \( \text{w-lim}_{n \to \infty} T^n x = x_0 \), then \( \lim_{n \to \infty} \sum_{m=1}^\infty a_{nm} T^m x = x_0 \) for every matrix \( (a_{nm}) \in \mathfrak{A}. \)

(\beta) For every uniformly bounded semigroup \( \{T_t: t > 0\} \) of linear operators on \( B \) for which \( T_t x \) is continuous on \( (0, \infty) \), if \( \text{w-lim}_{t \to \infty} T_t x = x_0 \), then \( \lim_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) = x_0 \) for every sequence \( (\mu_n) \in \mathfrak{A}. \)

The conclusion remains valid if “power bounded” and “uniformly bounded” in (\alpha) and (\beta) are both replaced by “contraction”.

\text{Proof. Let } x \text{ be a fixed element in } B, \text{ and assume that } (\alpha) \text{ holds for } x. \text{ Let } \{T_t: t > 0\} \text{ be a semigroup satisfying the hypotheses of } (\beta) \text{ and } (\mu_n) \in \mathfrak{A}. \text{ We shall show that } \lim_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) = x_0.

Let \( \varepsilon > 0. \) The continuity of \( T_t x \) on \([1, 2]\) implies that \( T_t x \) is uniformly continuous on \([1, 2]\). Thus there is a positive integer \( k \) such that if
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\[ g(t) = \sum_{j=1}^{k} I_{(1+(j-1)/k,1+j/k)}(t) \cdot T_{i+j/k}x, \]

then \( \|g(t) - T_t x\| < \varepsilon \) for \( t \in (1,2] \). Here \( I_A \) denotes the function that is 1 on \( A \), and 0 elsewhere. Set \( M = \sup_{t>0} \|T_t\| \), \( K = \sup_n \|\mu_n\| \), and \( I_i = (i,i+1] \). Since \( |\mu_n|(I_0) \to 0 \) by (2.1), we have

\[
\limsup_{n \to \infty} \left| \int_{I_i} T_t x \mu_n(dt) - \frac{1}{k} \int_{I_{i+1}} T_{i+(j-1)/k} x \mu_n(dt) \right|
\]

\[
= \limsup_{n \to \infty} \left| \int_{I_i} T_t x \mu_n(dt) - \sum_{i=0}^{\infty} \int_{I_{i+1}} (T_{i+x} - T_{i+j/k} g(t-i)) \mu_n(dt) \right|
\]

(2.3)

\[
\leq \limsup_{n \to \infty} \left[ M \cdot \|x\| \cdot |\mu_n|(I_0) + \sum_{i=0}^{\infty} \|T_i\| \cdot \sup_{i \in I_i} \|T_{i+x} - g(t)\| \cdot |\mu_n|(I_{i+1}) \right]
\]

\[
\leq M \cdot K \cdot \varepsilon.
\]

For each \( i \geq 0, 1 \leq j \leq k \), set \( I_{i,j} = (i + (j-1)/k, i+j/k] \). It follows from the definition of \( g(t) \) that for \( n \geq 1 \),

\[
\sum_{i=0}^{\infty} \int_{I_{i+1}} T_t g(t-i) \mu_n(dt) = \sum_{i=0}^{\infty} \sum_{j=1}^{k} \mu_n(I_{i+1,j}) \cdot T_{i+1+j/k} x
\]

(2.4)

where \( T = T_{i/k} \), and for \( m = (i+1)k + j, i \geq 0, 1 \leq j \leq k, a_{n,m} = \mu_n(I_{i+1,j}) \). It is easily checked that \( (a_{n,m}) \in \mathcal{U}_R \) since \( (\mu_n) \in \mathcal{U} \). Moreover, since \( \{T_t: t > 0\} \) is uniformly bounded and \( \lim_{t \to \infty} T_t x = x_0 \), we have that \( T \) is power bounded and \( \lim_{m \to \infty} T^m x = x_0 \). Thus it follows from (a) that \( \lim_{n \to \infty} \sum_{m \geq k} a_{n,m} T^m x = x_0 \). Together with (2.3) and (2.4), we obtain that

\[
\limsup_{n \to \infty} \left| \int_0^\infty T_t x \mu_n(dt) - x_0 \right| \leq M K \varepsilon.
\]

As \( \varepsilon > 0 \) is arbitrary, (\( \beta \)) holds.

It is clear that the second part of the theorem can be proved in the same way.

\[ \square \]

**Corollary 2.1.** Let \( B \) be a Banach space. Then (a)' implies (\( \beta \))':

(a)' For every power bounded linear operator \( T \) on \( B \), if \( \lim_{n \to \infty} T^n x \) exists for every \( x \in B \), then \( \lim_{n \to \infty} \sum_{m=k}^\infty a_{n,m} T^m x \) exists and is equal to \( \lim_{n \to \infty} T^n x \) for every \( (a_{n,m}) \in \mathcal{U}_R \).

(\( \beta \))' For every strongly continuous, uniformly bounded semigroup \( \{T_t: t > 0\} \) of linear operators on \( B \), if \( \lim_{t \to \infty} T_t x \) exists for every \( x \in B \), then \( \lim_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) \) exists for every sequence \( (\mu_n) \in \mathcal{U} \), and is equal to \( \lim_{t \to \infty} T_t x \).

The conclusion remains valid if "power bounded" and "uniformly bounded" are both replaced by "contraction".
Proof. Immediate from Theorem 2.1. □

We next show that the converse of statement (β) in Theorem 2.1 is valid in general Banach spaces.

Proposition 2.1. Let \( x \) be a fixed element in a Banach space \( B \), real or complex. Let \( \{T_t : t > 0\} \) be continuous linear operators on \( B \) such that the vector-valued function \( T_t x \) from \( (0, \infty) \) to \( B \) is continuous and \( \sup_{t>0} \|T_t x\| < \infty \). Then (b) implies (a):

(a) \( \text{w-lim}_{t \to \infty} T_t x \) exists.

(b) \( \lim_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) \) exists for every sequence \( \mu_n \in \mathcal{M} \).

Proof. We first consider the case where \( B \) is a real Banach space. Assume that (b) holds but (a) fails. Then there exists an \( x^* \in B^* \) such that \( h(t) = \langle T_t x, x^* \rangle \) diverges as \( t \to \infty \), where \( B^* \) is the dual space of \( B \). Since

\[
\sup_{t>0} |h(t)| \leq \sup_{t>0} \|x^*\| \|T_t x\| < \infty,
\]

\( h \) is bounded on \( (0, \infty) \). \( h(t) \) is also continuous on \( (0, \infty) \) since \( T_t x \) is. Thus, \( h(t) \) being divergent as \( t \to \infty \), there are constants \( a, \beta \) with \( a < \beta \), and a sequence \( (t_i) \) with \( t_i \uparrow \infty \), such that \( h(t_i) \geq \beta \) if \( i \) is odd, and \( h(t_i) \leq a \) if \( i \) is even. Set for \( n \geq 1 \),

\[
\mu_{2n} = \frac{1}{n} \sum_{k=1}^{n} \delta(t_{2k}), \quad \mu_{2n-1} = \frac{1}{n} \sum_{k=1}^{n} \delta(t_{2k-1}),
\]

where \( \delta(t) \) denotes the unit point mass at \( t \). Then \( (\mu_n) \in \mathcal{M} \), but

\[
\liminf_n \left( \int_0^\infty T_t x \mu_n(dt), x^* \right) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(t_{2k}) \leq a < \beta \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(t_{2k-1}) = \limsup_{n \to \infty} \left( \int_0^\infty T_t x \mu_n(dt), x^* \right).
\]

Hence \( (\int_0^\infty T_t x \mu_n(dt))_{n=1}^{\infty} \) does not converge weakly, and a fortiori, strongly.

If \( B \) is a complex Banach space, then either the real part or the imaginary part of \( h(t) \) diverges as \( t \to \infty \), and can be used to replace \( h(t) \) in the above argument. □

Remark 4. The vector-valued function \( T_t x \) in Proposition 3.1 may be replaced by any vector-valued function \( x(t) \) from \( (0, \infty) \) to \( B \) such that \( x(t) \) is continuous and bounded on \( (0, \infty) \).

3. We now apply the results in §2 to the Banach spaces \( L_p \) of a \( \sigma \)-finite measure space \( (X, \mathcal{A}, m), 1 \leq p < \infty \). An operator \( T \) on \( L_p \) is called positive if \( Tf \geq 0 \) whenever \( f \geq 0 \). Theorem 3.1 below strengthens the result of R. Sato mentioned in §1.

Theorem 3.1. Let \( \{T_t : t > 0\} \) be a contraction semigroup on \( L_2(X, \mathcal{A}, m) \), and let \( f \) be a fixed function in \( L_2 \) such that \( Tf \) is continuous on \( (0, \infty) \). Then conditions (a) and (b) are equivalent:
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(a) \( \lim_{t \to \infty} T_t f = f_0 \).
(b) \( \lim_n \int_0^\infty T_t f \mu_n (dt) = f_0 \) for every \( (\mu_n) \in \mathcal{A} \).

**Proof.** This follows from Theorem 1.1 in [6], Proposition 2.1 and Theorem 2.1. \( \square \)

**Theorem 3.2.** Let \( \{T_t ; t > 0\} \) be a strongly continuous contraction semigroup on \( L_1 (S, \mathcal{E}, m) \). Then conditions (A) and (B) are equivalent:

(A) For each \( f \in L_1 \), \( \lim_{t \to \infty} T_t f \) exists.
(B) For each \( f \in L_1 \), \( \lim_{n \to \infty} \int_0^\infty T_t f \mu_n (dt) \) exists for every \( (\mu_n) \in \mathcal{A} \), and is equal to \( \lim_{t \to \infty} T_t f \).

**Proof.** This follows from Theorem 1.3 in [6], Proposition 2.1, and Corollary 2.1. \( \square \)

**Theorem 3.3** Let \( \{T_t ; t > 0\} \) be a strongly continuous semigroup of positive contractions on \( L_p (X, \mathcal{E}, m) \), where \( p \) is fixed, \( 1 < p < \infty \). Then conditions (A) and (B) are equivalent:

(A) For each \( f \in L_p \), \( \lim_{t \to \infty} T_t f \) exists.
(B) For each \( f \in L_p \), \( \lim_{n \to \infty} \int_0^\infty T_t f \mu_n (dt) \) exists for each sequence \( (\mu_n) \in \mathcal{A} \), and is equal to \( \lim_{n \to \infty} T_t f \).

**Proof.** We observe that the conclusions in Theorem 2.1 and Corollary 2.1 remain valid if \( B = L_p (X, \mathcal{E}, m) \), and “power bounded” and “uniformly bounded” in (a) and (b) are both replaced by “positive contraction”. Theorem 3.3 now follows from Theorem 1.4 in [3] and Proposition 2.1. \( \square \)

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**References**


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