WEAK CONVERGENCE OF SEMIGROUPS IMPLIES STRONG CONVERGENCE OF WEIGHTED AVERAGES

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Abstract. For a fixed \( p \), \( 1 \leq p < \infty \), let \( \{ T_t : t > 0 \} \) be a strongly continuous semigroup of positive contractions on \( L_p \) of a \( \sigma \)-finite measure space. We show that weak convergence of \( \{ T_t : t > 0 \} \) in \( L_p \) is equivalent with the strong convergence of the weighted averages \( \int_0^\infty T_t f \mu_n(dt) \) \( (n \to \infty) \) for every \( f \in L_p \) and every sequence \( \{ \mu_n \} \) of signed measures on \( (0, \infty) \), satisfying \( \sup_n \| \mu_n \| < \infty \); \( \lim_n \mu_n(0, \infty) = 1 \); and for each \( d > 0 \), \( \limsup_{n \to \infty} \mu_n(c,c+d] = 0 \). The positivity assumption is not needed if \( p = 1 \) or \( 2 \). We show that such a result can be deduced—not only in \( L_p \) but in general Banach spaces—from the corresponding discrete parameter version of the theorem.

In recent years, various authors have studied the relations between weak and strong operator convergence: Blum-Hanson [4], Hanson-Pledger [7], Lin [9], Akcoglu-Sucheston [1], Jones-Kuftinec [8], Fong-Sucheston [6], and very recently Akcoglu-Sucheston [2], [3], who proved the theorem for positive \( L_p \)-contractions, \( 1 < p < \infty \). This theorem [1], [6], [3] states that if \( T \) is a positive contraction on \( L_p \) of a \( \sigma \)-finite measure space \( (X, \mathcal{E}, \mu) \), where \( p \) is fixed and \( 1 < p < \infty \), then weak-lim \( n \to \infty \) \( T^n f (w \lim_n T^n f) \) exists for each \( f \in L_p \) if, and only if, \( \lim_n \sum_{m=1}^\infty a_{nm} T^m f \) exists for every \( f \in L_p \) and every matrix \( \{ a_{nm} \} \) with real entries satisfying

\[
(1.1) \quad \sup_n \sum m |a_{nm}| < \infty; \lim_n \sum m a_{nm} = 1; \lim_{n \to \infty} \max_m |a_{nm}| = 0.
\]

It has also been shown that the positivity assumption is not needed if \( p = 1 \) or 2 [1], [6]. The problem of whether or not positivity is needed for \( p \neq 1 \) or 2 is still open. Matrices satisfying (1.1) were introduced in ergodic context in [6] and have been called uniformly regular; we denote the class of all uniformly regular matrices by \( \mathcal{A}_R \). Intuitively, a matrix \( \{ a_{nm} \} \in \mathcal{A}_R \) if and only if it is properly averaging, in the sense that the masses \( a_{nm} \) spread as \( n \to \infty \).

A semigroup \( \{ T_t : t > 0 \} \), \( T_t T_s = T_{t+s} \), of linear operators on a Banach space \( B \) is called strongly continuous if for each \( x \in B \) and each \( s > 0 \), \( \lim_{t \to 0} \| T_t x - T_s x \| = 0 \). R. Sato [10] recently obtained the following continuous parameter version of the strong ergodic theorem: For a fixed function \( f \) in \( L_2 \) and a strongly continuous semigroup \( \{ T_t : t > 0 \} \) of contractions on \( L_2 \), \( w \)-lim \( n \to \infty \) \( T_t f \) \( = f_0 \) implies \( \lim_{n \to \infty} \int_0^\infty a_n(t) T_t f dt = f_0 \) for every sequence \( \{ a_n \} \) of nonnegative, Lebesgue integrable functions on \( (0, \infty) \) satisfying...
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\[ \int_0^\infty a_n(t) = 1 \text{ and } \lim_{n \to \infty} \|a_n\|_\infty = 0. \] In this note, we show that a stronger result can be deduced—not only in \( L_2 \), but in general Banach spaces—from the corresponding discrete version of the theorem (§2). In §3, we obtain as corollaries the continuous parameter version of the Akcoglu-Sucheston theorem for the Banach spaces \( L_p(X, \mathcal{F}, m) \), \( 1 \leq p < \infty \).

2. A linear operator \( T \) on a Banach space \( B \) is called \textit{power-bounded} if \( \sup_n \| T^n \| < \infty \); a semigroup \( \{ T_t : t > 0 \} \) of linear operators on \( B \) is called \textit{uniformly bounded} if \( \sup_{t > 0} \| T_t \| < \infty \). The \textit{total variation} of a signed measure \( \mu \) is denoted by \( |\mu| \). We denote by \( \mathfrak{A} \) the family of all sequences \( (\mu_n) \) of signed measures on the \( \sigma \)-algebra of Lebesgue measurable subsets of (0, \( \infty \)) satisfying

\[
\sup_n \| \mu_n \| < \infty ; \quad \lim_{n \to \infty} \mu_n(0, \infty) = 1 ;
\]

\( (2.1) \)

\[
\lim_{n \to \infty} \sup_{c > 0} |\mu_n|(c, c + d] = 0 \quad \text{for each} \quad d > 0.
\]

\textbf{Remark 1.} If \( (a_n) \) is a sequence of Lebesgue integrable functions on (0, \( \infty \)) satisfying

\[
\sup_n \int_0^\infty |a_n(t)| \, dt < \infty ; \quad \lim_{t \to 0} \int_0^\infty a_n(t) \, dt = 1 ;
\]

\( (2.2) \)

\[
\lim_{n \to \infty} \sup_{c > 0} \int_c^{c+d} |a_n(t)| \, dt = 0
\]

for each \( d > 0 \), and if we set \( d\mu_n = a_n \, dt \), then \( (\mu_n) \in \mathfrak{A} \). We note that sequences \( (a_n) \) satisfying (2.2) include those considered by Sato in [10].

\textbf{Remark 2.} Let \( \delta(t) \) denote the unit point mass at \( t \). If \( t_m > 0, t_m \to \infty, (a_{nm}) \in \mathfrak{A}_R \), and if we set \( \mu_n = \sum_m a_{nm} \delta(t_m) \), then \( (\mu_n) \in \mathfrak{A} \).

\textbf{Remark 3.} If \( x(t) \) is a bounded continuous function from (0, \( \infty \)) to a Banach and if we set \( d\mu_n = a_n \, dt \), then \( (\mu_n) \in \mathfrak{A} \). We note that sequences \( (a_n) \) satisfying (2.2) include those considered by Sato in [10].

\[
\left\| \int_0^\infty x(t) \mu(dt) \right\| \leq \left( \sup_{t > 0} \| x(t) \| \right) \cdot \| \mu \| \quad (\text{cf. [5]}).
\]

\textbf{Theorem 2.1.} Let \( x \) be a fixed element in a Banach space \( B \), real or complex. Then \( (\alpha) \) implies \( (\beta) \):

\( (\alpha) \) Every power bounded linear operator \( T \) on \( B \), if \( \text{w-lim}_{n \to \infty} T^n x = x_0 \), then \( \lim_{n \to \infty} \sum_{m=1}^\infty a_{nm} T^m x = x_0 \) for every matrix \( (a_{nm}) \in \mathfrak{A}_R \).

\( (\beta) \) For every uniformly bounded semigroup \( \{ T_t : t > 0 \} \) of linear operators on \( B \) for which \( T_t x \) is continuous on (0, \( \infty \)), if \( \text{w-lim}_{t \to \infty} T_t x = x_0 \), then \( \lim_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) = x_0 \) for every sequence \( (\mu_n) \in \mathfrak{A} \).

The conclusion remains valid if "power bounded" and "uniformly bounded" in \( (\alpha) \) and \( (\beta) \) are both replaced by "contraction".

\textbf{Proof.} Let \( x \) be a fixed element in \( B \), and assume that \( (\alpha) \) holds for \( x \). Let \( \{ T_t : t > 0 \} \) be a semigroup satisfying the hypotheses of \( (\beta) \) and \( (\mu_n) \in \mathfrak{A} \). We shall show that \( \text{lim}_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) = x_0 \).

Let \( \varepsilon > 0 \). The continuity of \( T_t x \) on [1, 2] implies that \( T_t x \) is uniformly continuous on [1,2]. Thus there is a positive integer \( k \) such that if
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\[ g(t) = \sum_{j=1}^{k} I_{1+1+(j-1)/k,1+j/k}(t) \cdot T_{i+j/k} x, \]

then \( \|g(t) - T_t x\| < \varepsilon \) for \( t \in (1,2] \). Here \( I_A \) denotes the function that is 1 on \( A \), and 0 elsewhere. Set \( M = \sup_{t>0} \|T_t\| \), \( K = \sup_n \|\mu_n\| \), and \( I_i = (i,i+1] \). Since \( |\mu_n|(I_0) \to 0 \) by (2.1), we have

\[
\limsup_{n \to \infty} \left| \int_0^\infty T_t x u_n(dt) - \sum_{i=0}^{\infty} \int_{I_{i+1}} (T_t x - T_t g(t - i)) u_n(dt) \right|
\]

\[
\leq \limsup_{n \to \infty} \left( M \cdot \|x\| \cdot |\mu_n|(I_0)
\right.
\]

\[
+ \sum_{i=0}^{\infty} \|T_i\| \cdot \sup_{i \in I_i} \|T_i x - g(t)\| \cdot |\mu_n|(I_{i+1}) \right)
\]

\[
\leq M \cdot K \cdot \varepsilon.
\]

For each \( i \geq 0, 1 \leq j \leq k \), set \( I_{i,j} = (i + (j - 1)/k, i + j/k] \). It follows from the definition of \( g(i) \) that for \( n \geq 1 \),

\[
\sum_{i=0}^{\infty} \int_{I_{i+1}} T_t g(t - i) u_n(dt) = \sum_{i=0}^{\infty} \sum_{j=1}^{k} \mu_n(I_{i+1,j}) \cdot T_{i+1+j/k} x
\]

\[
= \sum_{m=k+1}^{\infty} a_{n,m} T^m x,
\]

where \( T = T_{1/k} \), and for \( m = (i + 1)k + j, i \geq 0, 1 \leq j \leq k, a_{n,m} = \mu_n(I_{i+1,j}) \). It is easily checked that \( (a_{n,m}) \in \mathbb{R}^R \) since \( (\mu_n) \in \mathbb{R} \). Moreover, since \( \{T_t: t > 0\} \) is uniformly bounded and \( w\text{-lim}_{t \to \infty} T_t x = x_0 \), we have that \( T \) is power bounded and \( w\text{-lim}_{m \to \infty} T^m x = x_0 \). Thus it follows from (a) that \( \lim_{n \to \infty} \sum_{m=k}^{\infty} a_{n,m} T^m x = x_0 \). Together with (2.3) and (2.4), we obtain that

\[
\limsup_{n \to \infty} \left| \int_0^\infty T_t x u_n(dt) - x_0 \right| \leq M \varepsilon K.
\]

As \( \varepsilon > 0 \) is arbitrary, (\( \beta \)) holds.

It is clear that the second part of the theorem can be proved in the same way. \( \square \)

**Corollary 2.1.** Let \( B \) be a Banach space. Then (a)' implies (\( \beta \))':

(a)' For every power bounded linear operator \( T \) on \( B \), if \( w\text{-lim}_{n \to \infty} T^n x \) exists for every \( x \in B \), then \( \lim_{n \to \infty} \sum_{m=k}^{\infty} a_{n,m} T^m x \) exists and is equal to \( w\text{-lim}_{n \to \infty} T^n x \) for every \( (a_{n,m}) \in \mathbb{R}^R \).

(\( \beta \))' For every strongly continuous, uniformly bounded semigroup \( \{T_t: t > 0\} \) of linear operators on \( B \), if \( w\text{-lim}_{t \to \infty} T_t x \) exists for every \( x \in B \), then \( \lim_{n \to \infty} \int_0^\infty T_t x u_n(dt) \) exists for every sequence \( (\mu_n) \in \mathbb{R} \), and is equal to \( \lim_{n \to \infty} T_t x \).

The conclusion remains valid if "power bounded" and "uniformly bounded" are both replaced by "contraction".

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Proof. Immediate from Theorem 2.1. □

We next show that the converse of statement (β) in Theorem 2.1 is valid in general Banach spaces.

Proposition 2.1. Let \( x \) be a fixed element in a Banach space \( B \), real or complex. Let \( \{T_t: t > 0\} \) be continuous linear operators on \( B \) such that the vector-valued function \( T_t x \) from \( (0, \infty) \) to \( B \) is continuous and \( \sup_{t>0} \|T_t x\| < \infty \). Then (b) implies (a):

(a) \( \text{w-lim}_{t \to \infty} T_tx \) exists.
(b) \( \text{lim}_{n \to \infty} \int_0^\infty T_t x \mu_n(dt) \) exists for every sequence \( (\mu_n) \in \mathfrak{A} \).

Proof. We first consider the case where \( B \) is a real Banach space. Assume that (b) holds but (a) fails. Then there exists an \( x^* \in B^* \) such that \( h(t) = \langle T_t x, x^* \rangle \) diverges as \( t \to \infty \), where \( B^* \) is the dual space of \( B \). Since

\[
\sup_{t>0} |h(t)| \leq \sup_{t>0} \|x^*\| \|T_t x\| < \infty,
\]

\( h \) is bounded on \( (0, \infty) \). \( h(t) \) is also continuous on \( (0, \infty) \) since \( T_t x \) is. Thus, \( h(t) \) being divergent as \( t \to \infty \), there are constants \( \alpha, \beta \) with \( \alpha < \beta \), and a sequence \( (t_i) \) with \( t_i \uparrow \infty \), such that \( h(t_i) \geq \beta \) if \( i \) is odd, and \( h(t_i) \leq \alpha \) if \( i \) is even. Set for \( n > 1 \),

\[
\mu_{2n} = \frac{1}{n} \sum_{k=1}^{n} \delta(t_{2k}), \quad \mu_{2n-1} = \frac{1}{n} \sum_{k=1}^{n} \delta(t_{2k-1}),
\]

where \( \delta(t) \) denotes the unit point mass at \( t \). Then \( (\mu_n) \in \mathfrak{A} \), but

\[
\liminf_n \left\langle \int_0^\infty T_t x \mu_n(dt), x^* \right\rangle = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(t_{2k})
\]

\[
\leq \alpha < \beta \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(t_{2k-1})
\]

\[
= \limsup_{n \to \infty} \left\langle \int_0^\infty T_t x \mu_n(dt), x^* \right\rangle.
\]

Hence \( \left( \int_0^\infty T_t x \mu_n(dt) \right)_{n=1}^\infty \) does not converge weakly, and \emph{a fortiori}, strongly.

If \( B \) is a complex Banach space, then either the real part or the imaginary part of \( h(t) \) diverges as \( t \to \infty \), and can be used to replace \( h(t) \) in the above argument. □

Remark 4. The vector-valued function \( T_t x \) in Proposition 3.1 may be replaced by any vector-valued function \( x(t) \) from \( (0, \infty) \) to \( B \) such that \( x(t) \) is continuous and bounded on \( (0, \infty) \).

3. We now apply the results in §2 to the Banach spaces \( L_p \) of a \( \sigma \)-finite measure space \( (X, \mathcal{A}, m) \), \( 1 \leq p < \infty \). An operator \( T \) on \( L_p \) is called positive if \( Tf \geq 0 \) whenever \( f \geq 0 \). Theorem 3.1 below strengthens the result of R. Sato mentioned in §1.

Theorem 3.1. Let \( \{T_t: t > 0\} \) be a contraction semigroup on \( L_2(X, \mathcal{A}, m) \), and let \( f \) be a fixed function in \( L_2 \) such that \( Tf \) is continuous on \( (0, \infty) \). Then conditions (a) and (b) are equivalent:
(a) \( \lim_{t \to \infty} T_t f = f_0 \).

(b) \( \lim_{n} \int_0^\infty T_t f \mu_n (dt) = f_0 \) for every \( (\mu_n) \in \mathfrak{A} \).

**Proof.** This follows from Theorem 1.1 in [6], Proposition 2.1 and Theorem 2.1. \( \square \)

**Theorem 3.2.** Let \( \{T_t : t > 0\} \) be a strongly continuous contraction semigroup on \( L_1 (S, \mathcal{A}, m) \). Then conditions (A) and (B) are equivalent:

(A) For each \( f \in L_1 \), \( \lim_{t \to \infty} T_t f \) exists.

(B) For each \( f \in L_1 \), \( \lim_{n} \int_0^\infty T_t f \mu_n (dt) \) exists for every \( (\mu_n) \in \mathfrak{A} \), and is equal to \( \lim_{t \to \infty} T_t f \).

**Proof.** This follows from Theorem 1.3 in [6], Proposition 2.1, and Corollary 2.1. \( \square \)

**Theorem 3.3** Let \( \{T_t : t > 0\} \) be a strongly continuous semigroup of positive contractions on \( L_p (X, \mathcal{A}, m) \), where \( p \) is fixed, \( 1 < p < \infty \). Then conditions (A) and (B) are equivalent:

(A) For each \( f \in L_p \), \( \lim_{t \to \infty} T_t f \) exists.

(B) For each \( f \in L_p \), \( \lim_{n} \int_0^\infty T_t f \mu_n (dt) \) exists for each sequence \( (\mu_n) \in \mathfrak{A} \), and is equal to \( \lim_{t \to \infty} T_t f \).

**Proof.** We observe that the conclusions in Theorem 2.1 and Corollary 2.1 remain valid if \( B = L_p (X, \mathcal{A}, m) \), and “power bounded” and “uniformly bounded” in (a) and (b) are both replaced by “positive contraction”. Theorem 3.3 now follows from Theorem 1.4 in [3] and Proposition 2.1. \( \square \)

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**References**


