SYMMETRIC OVERMAPS

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ABSTRACT. We prove periodicity theorems for the degrees of fibre-preserving maps of sphere bundles, and of projective space bundles.

This note is our first on the subject of fibre-preserving maps, called \textit{overmaps}, and comes from [6]. I wish to thank my supervisor, Professor I. M. James, for encouragement and for considerable help with the exposition.

Let $M, M'$ be connected compact oriented $q$-manifolds, where $q \geq 1$. For $K \geq 1$, let a transitive permutation group $\Gamma$ on $K$ letters permute the factors of $M^K$. A map from $M^K$ is \textit{symmetric} when it is constant on the orbits of $\Gamma$. The \textit{degree} of a symmetric map is the Brouwer degree of its restriction $M \to M'$ to a factor.

Constant maps are symmetric and, if $K = 1$, all maps from $M$ to $M'$ are symmetric. If $M$ is a rational cohomology sphere then, by [2], a necessary condition for there to be a symmetric map $M^K \to M'$ of nonzero degree is that $q$ be odd or $K = 1$.

Let $E$, $E'$ be oriented fibre bundles over a path-connected space $B$ with fibres $M, M'$. We denote the fibre product $E \times_B E \times_B \cdots E$ ($K$ factors) by $E^{(K)}$. An overmap $E^{(K)} \to E'$ is \textit{symmetric of degree} $m$ when its restriction to fibres is a symmetric map of degree $m$. In particular, if $K = 1$, all overmaps from $E$ to $E'$ are symmetric.

Let a group $G$ act fibrewise on $E$, with the product action on $E^{(K)}$, and fibrewise orthogonally on an oriented orthogonal $q$-sphere bundle $F$ over $B$. When $q$ is odd we orient the real projective $q$-space bundle $PF$ associated with $F$, and we let $G$ act, so that the identification overmap $h: F \to PF$ is $G$-invariant of degree 2.

\textbf{Theorem 1.} Let $q$ be odd, let $E'$ be $F$ or $PF$, and let $n$ be the degree of a $G$-invariant symmetric overmap from $E^{(K)}$ to $E'$. There is an integer $\alpha_K(E, E') \geq 0$ such that there is a $G$-invariant symmetric overmap $E^{(K)} \to E'$ of degree $m$ if and only if $m \equiv n \mod \alpha$.

Taking $E = E'$, $K = n = 7$ in Theorem 1, we obtain the following result.

\textbf{Corollary 2.} Let $q$ be odd, and let $E'$ be $F$ or $PF$. There is an integer $\alpha(E') \geq 0$ such that there is a $G$-invariant overmap of degree $m$ from $E'$ to itself if and only if $m \equiv 1 \mod \alpha$.

Received by the editors April 16, 1975 and, in revised form, September 3, 1975.


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In [7] we take $G$ to be trivial, and we show that as $E'$ varies $a(E')$ takes all nonnegative integer values. In general, $a(PF) \neq a(F)$. For example, let $G$ be trivial, and let $F$ be the Whitney multiple $(q + 1)h$ of the identification $h: S^n \to RP^n$, where $S^n$, $RP^n$ are the $n$-sphere, projective real $n$-space. Then $a(F) = 1, 2$ according as $n \leq q, n > q$. However, $PF$ is trivial and, by Corollary 2, $a(PF) = 1$.

Let the fibre bundles $E$, $E'$ be ex-spaces [4]. Thus $E$, $E'$ are equipped with cross-sections $s, s'$ which we suppose $G$-invariant. Also an overmap $f: E \to E'$ is an ex-map when $fs = s'$. We regard $E^{(K)}$ as an ex-space by means of the cross-section $s \times_B s \times_B \cdots s$. Let the sphere bundle $F$ be an ex-space, and regard $PF$ as an ex-space by requiring $h: F \to PF$ to be an ex-map.

**Theorem 3.** Let $q$ be odd, and let $E'$ be $F$ or $PF$. There is an integer $\beta_K(E, E')$ such that there is a $G$-invariant symmetric ex-map $E^{(K)} \to E'$ of degree $m$ if and only if $m = 0 \mod \beta$.

Taking $E = E'$, $K = 1$ in Theorem 3, we obtain the following result.

**Corollary 4.** Let $q$ be odd, and let $E'$ be $F$ or $PF$. There are $G$-invariant ex-maps of all degrees from $E'$ to itself.

When $B$ is a connected compact oriented manifold, so are $E$, $E'$, and the degree of an overmap from $E$ to $E'$ is its Brouwer degree. Hence, and by fibre-suspension, Corollary 4 generalizes [8, 1.5].

Let $G$ be trivial, and let $B$ be a connected finite CW-complex. When $q$ is odd, and $E'$ is $F$ or $PF$, then $\beta_K(E, E')$ depends on the vertical homotopy classes of $s, s'$, whereas $a_K(E, E')$ evidently does not. For example, let $E = E' = B \times S^q$, $B = S^r$, and let $s, s'$ correspond to $0, \nu \in \pi_r S^q$. Then $\beta_1(E, E')$ is the order of the Whitehead product $[\nu, \iota_q]$, where $\iota_q$ generates $\pi_q S^q$. However, $a_1(E, E') = 1$.

Because of the main result of [1], the argument of [3, §2] also applies to real projective spaces. Corollary 4 then allows us to argue fibrewise, proving the following generalization of [3, 2.3].

**Corollary 5.** Let $q$ be odd, and let $E'$ be $F$ or $PF$. Then $\beta_K(E, E')$ is positive. Further, no prime factor of $\beta_K(F, E')$ or of $\beta_K(PF, E')$ exceeds $K$.

Let $B$ be a point. Taken with Corollary 5, Theorem 3 generalizes [3, 1.2] to include symmetric maps of projective spaces. By [5] $\beta_2(S^q, S^q)$ is $2^{(q+1)/2}$ or $2^{(q-1)/2}$ according as $q \equiv 3, 5$ or $q \equiv 1, 7 \mod 8$.

To prove Theorems 1, 3, let $O(q + 1)$ denote the group of orthogonal transformations of $S^q$. Let $s, t$ be integers, and define an $O(q + 1)$-map $k_i: S^q \times S^q \to S^q$ as follows.

$$\begin{align*}
k'_i(x, y) &= (x \sin(1 - i)\theta + y \sin t\theta) \cosec \theta, \\
k'_i(x, x) &= x, \quad k'_i(x, -x) = (-1)^i x,
\end{align*}$$

where $x, y \in S^q, x \neq \pm y$, and $0 < \theta < \pi$ is chosen so that $\cos \theta$ is the Euclidean inner product $(x \cdot y)$.
For $x, y \in S^q$ we have the following identities.

(2) $k'_i(x, y) = k'_{1-i}(y, x),$

(3) $k'_s(x, y) = k'_s(x, k'_s(x, y)),$

(4) $k'_i(x, -y) = (-1)^ik'_i(x, y),$

(5) $k'_{-1}(x, y) = T_{q+2}(x)(j'_{-1}(y)),$

where $T_{q+2}: S^q \to O(q + 1)$ is the characteristic map [9, §23.4] for the tangent bundle to $S^{q+1}$, and where $j'_{-1}: S^q \to S^q$ is the suspension of the antipodal map on the hyperplane orthogonal to $p^q = (0, 0, \ldots, 1) \in S^q$.

We may also describe $k'_i$ as follows. Given $x, y \in S^q$, let $\theta$ be the distance along a geodesic from $x$ to $y$. On this geodesic, and at distance $t\theta$ from $x$, we have $k'_i(x, y)$. From this description, or by induction on $t$ using (2), (3), (5), $k'_i$ is continuous. We denote $k'_i|\{p^q\} \times S^q$ by $j_i: S^q \to S^q$.

(6) According as $q$ is odd or even, $j_i$ has degree $t$ or $(1 + (-1)^{t-1})/2$.

To prove (6), note that if $t > 0$ then $j_t = 1 + a + 1 + \cdots$ ($t$ summands), where $1, a$ denote the identity, the antipodal map on $S^q$, and where '+' means head to tail addition along the $p^q$ axis. Since $a$ has degree $(-1)^{q+1}$ this proves (6) for $t > 0$. But $j_{-t} = j_{-1}j_t$ by (3), and $j_{-1}, j_0$ have degrees $(-1)^q, 0$. This completes the proof of (6).

By (4), (2), $k'_i$ respects the identification $h: S^q \to RP^q$, and therefore projects to an $O(q + 1)$-map $RP^q \times RP^q \to RP^q$ which we also refer to as $k'_i$. Let $q$ be odd. By (6), (2), and since $h$ is of degree 2, we have the following assertion.

(7) The restriction of $k'_i$ to the first, second factor has degree $1 - t, t$.

In the situation of Theorem 1, $k'_i$ extends from fibres to a $G$-invariant overmap $g: E' \times_B E' \to E'$. If $E'$ is an ex-space then $g$ is an ex-map, since $k'_i(x, x) = x$ by (1).

Let $\Delta: E^{(K)} \to E^{(K)} \times_B E^{(K)}$ be the diagonal overmap. Given $G$-invariant symmetric overmaps $f_i: E^{(K)} \to E'$ of degrees $m_i$

(8) $i = 1, 2$, the composite $g(f_2 \times_B f_1)\Delta$ is a $G$-invariant symmetric overmap of degree $tm_1 + (1 - t)m_2$.

Taken with the following remark, (8) proves Theorem 1.

Let $A$ be a nonempty set of integers such that, if $m_1, m_2 \in A$

(9) then, for all integers $t, tm_1 + (1 - t)m_2 \in A$. Then, for some integers $n, a, A = \{m: m \equiv n \text{ mod } a\}$.

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In the situation of Theorem 3, we may read ‘ex-map’ for ‘overmap’ in (8). Taken with (9), this proves Theorem 3.

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