A NOTE ON SOME PROPERTIES OF \( A \)-FUNCTIONS

H. SARBADHIKARI

Abstract. This note deals with \((M, \ast)\) functions for various families \(M\). It is shown that if \(M\) is the family of Borel sets of additive class \(\alpha\) on a metric space \(X\), then \((M, \ast)\) functions are just the functions of the form \(\sup_y g(x, y)\) where \(g : X \times R \to R\) is continuous in \(y\) and of class \(\alpha\) in \(x\). If \(M\) is the class of analytic sets in a Polish space \(X\), then the \((M, \ast)\) functions dominating a Borel function are just the functions \(\sup_y g(x, y)\) where \(g\) is a real valued Borel function on \(X^2\). It is also shown that there is an \(A\)-function \(f\) defined on an uncountable Polish space \(X\) and an analytic subset \(C\) of the real line such that \(f^{-1}(C) \not\in \sigma\)-algebra generated by the analytic sets on \(X\).

1. Introduction. Let \(X\) be any set and \(M, N\) be classes of subsets of \(X\). Following Hausdorff, we call a real valued function \(f\) on \(X\) a function of class \((M, \ast)\) if \(\{x : f(x) > c\}\) is in \(M\) for every \(c\). If \(\{x : f(x) \geq c\}\) is in \(N\) for every \(c\), \(f\) is said to be of class \((\ast, N)\). Set \((M, N) = (M, \ast) \cap (\ast, N)\).

If \(X\) is a metric space and \(M\) is the family of sets of additive Borel class \(\alpha\), then functions of class \((M, \ast)\) are called \(\alpha\)-functions; if \(X\) is Polish and \(M\) is the family of analytic sets, they are called \(A\)-functions. We shall prove the following theorems:

**Theorem 1.** Let \(f\) be a real valued function on a metric space \(X\). Then \(f\) is an \(\alpha\)-function if, and only if, there is a real valued function \(g\) defined on \(X \times R\), where \(R\) is the real line, such that \(g(x, y)\) is a continuous function of \(y\) for fixed \(x\), is of class \(\alpha\) in \(x\) for fixed \(y\) and \(f(x) = \sup_y g(x, y)\).

**Theorem 2.** Let \(X\) be a Polish space and let \(f\) be a real valued function on \(X\) which is bounded below. Then \(f\) is an \(A\)-function if, and only if, there is a real valued Borel function \(g\) on \(X^2\) such that \(f(x) = \sup_y g(x, y)\).

**Theorem 3.** Let \(A\) be the \(\sigma\)-algebra generated by analytic sets on an uncountable Polish space \(X\). There is an \(A\)-function \(f\) on \(X\) and an analytic subset \(C\) of the real line such that \(f^{-1}(C) \not\in A\).

Theorem 3 answers in the negative a question raised by David Blackwell.

2. Proof of Theorem 1. We define a complete ordinary function system on a set \(X\) as a system \(F\) of real valued functions on \(X\) satisfying:
(a) Every constant function is in $F$.
(b) If $f, g \in F$, then $\max (f, g), \min (f, g), f \pm g, f \cdot g \in F$. If $g$ does not vanish anywhere, then $f/g \in F$.
(c) If $f_n \in F$ for all $n$ and $f_n$ converges uniformly to $f$, then $f \in F$.

We first prove the following:

**Theorem 4.** Let $F$ be a complete ordinary function system on a set $X$. Let $P$, $Q$ be the families of sets $(x: h(x) > c), (x: h(x) \geq c)$, for $h \in F$ and $c$ real, respectively. $f \in (P, *)$ if, and only if, there is a real valued function $g$ defined on $X \times R$ such that $g(x, y)$

(a) is continuous in $y$ for fixed $x$,
(b) is in $F$ for fixed $y$, and
(c) $\sup_y g(x, y) = f(x)$.

**Proof.** Suppose $g(x, y)$ is a function on $X \times R$ satisfying conditions (a) and (b) and suppose $\sup_y g(x, y)$ exists and is $f(x)$. Let $c$ be any real number. Then $f(x) > c \iff \exists y \{g(x, y) > c\} \iff \exists y \{y \text{ is rational and } g(x, y) > c\}$, since $g(x, y)$ is continuous in $y$. Thus

$$\{x: f(x) > c\} = \bigcup_{r \text{ rational}} \{x: g(x, r) > c\}.$$ 

For fixed $r$, $g(x, r) \in F$ and hence $\{x: g(x, r) > c\} \in P$. Now as $P$ is closed under countable unions (cf. [1]), $\{x: f(x) > c\} \in P$.

Conversely, suppose $f \in (P, \ast)$. It is shown in [1] that there is an increasing sequence $\{f_n\}$ in $F$ which converges to $f$. Define $g$ on $X \times R$ by $g(x, y) = (f_{n+1}(x) - f_n(x))(|y| - n) + f_n(x)$ for $|y| \in [n, n + 1]$. It is easy to see that $g$ is well defined for all $(x, y)$ and satisfies (a) and (b). As $f_n(x) \leq g(x, y) \leq f_{n+1}(x)$ for $|y| \in [n, n + 1]$ and $\sup_n f_n(x) = f(x)$, $\sup_y g(x, y) = f(x)$.

Theorem 1 follows from Theorem 4 and the following:

**Lemma.** Let $F$ be the family of all functions of class $\alpha$ on a Polish space $X$. Then $F$ is a complete ordinary function system and the sets of the form $(x: f(x) > c), f \in F, c$ real, are just the sets of additive Borel class $\alpha$.

**Proof.** It is shown in [3] that $F$ forms a complete ordinary function system.

Any set of the form $(x: f(x) > c), f \in F, c$ real, is clearly of additive Borel class $\alpha$. Let $A$ be any set of additive Borel class $\alpha$. If $\alpha = 0$, $A$ is a cozero set and hence $A = \{x: f(x) > 0\}$ for some continuous function $f$. If $\alpha > 0$, then we can write $A = \bigcup_{n=1}^{\infty} A_n$ where the $A_n$'s are ambiguous of class $\alpha$. Let $f(x) = \sum_{n=1}^{\infty} 2^{-n} I_{A_n}(x)$ where $I_{A_n}$ denotes the indicator function of $A_n$. As $I_{A_n}$ is of class $\alpha$, $f$ is of class $\alpha$ and $A = \{x: f(x) > 0\}$.

3. **Proof of Theorem 2.** If $f(x) = \sup_y g(x, y)$ where $g$ is Borel measurable, it is shown in [3] that $f$ is an $A$-function. For this, $f$ need not be bounded below.

Let $f$ be an $A$-function on $X$ such that $f(x) > a$ for a fixed real number $a$. Without loss of generality, we take $X = R$. Let $(\eta_n)$ enumerate all rationals. Let $A = \{(x, y): f(x) > y\}$. Then $A = \bigcup_{n} f(\eta_n)(x) > \eta_n > y)$ and hence is analytic. Let $B \subset R^3$ be a Borel set such that $A = \text{projection of } B$ i.e. $(x, y) \in A \iff \exists z((x, y, z) \in B)$. Let $k: R^3 \to R^3$ be defined by

$$k(x, y, z) = \begin{cases} (x, y, z) & \text{if } (x, y, z) \in B, \\ (a, a, a) & \text{otherwise}. \end{cases}$$

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Then, as \( k \) is Borel measurable so is \( \tau_2 k \) where \( \tau_2 \) denotes projection to the second coordinate and

\[
\tau_2 k(x, y, z) = \begin{cases} 
  y & \text{if } (x, y, z) \in B,
  a & \text{otherwise}.
\end{cases}
\]

Thus \( \sup_{(y, z)} \tau_2 k(x, y, z) = \sup_{(y, z)} \{ y : y < f(x) \} \cup \{ a \} = f(x) \). Let \( \phi \) be a Borel isomorphism from \( R \) onto \( R^2 \). Let \( h : R^2 \to R^3 \) be defined by \( h(x, y) = (x, \phi(y)) \) and let \( g(x, y) = \tau_2 k h(x, y) \). Then \( g \) is Borel measurable and \( f(x) = \sup_y \tau_2 k(x, \phi(y)) = \sup_y g(x, y) \).

**Remark.** It is easy to see that Theorem 2 holds even if the condition "\( f \) is bounded below" is replaced by "\( f \) dominates a Borel function". Thus an \( A \)-function is of the form \( \sup_y g(x, y) \) for some Borel measurable \( g \) if, and only if, it dominates a Borel function. Equivalently, every \( A \)-function is of the form \( \sup_y g(x, y) \) for some Borel measurable \( g \) if, and only if, given an ascending sequence of analytic sets \( \{ A_n \} \) such that \( \bigcup_{n=1}^{\infty} A_n = X \), there is an ascending sequence \( \{ B_n \} \) of Borel sets such that \( B_n \subset A_n \) and \( \bigcup_{n=1}^{\infty} B_n = X \). However, we do not know if this condition always holds.

4. **Proof of Theorem 3.** In \( X \), we put \( S_0 = \) the family of open sets, \( B_0 = \sigma(S_0) \) and, for \( 0 < \alpha < \omega_1 \), \( S_\alpha = \sigma(\bigcup_{\beta < \alpha} S_\beta) \) and \( B_\alpha = \sigma(S_\alpha) \) where, for any family of sets \( G \), \( \sigma(G) \) denotes the \( \sigma \)-algebra generated by \( G \) and \( \mathcal{A}(G) \) denotes the smallest family containing \( G \) and closed under operation \( A \).

We call \( (S_\alpha, *) \) functions \( S_\alpha \)-functions. Theorem 3 is obtained from the following more general theorem by putting \( \alpha = 1 \).

**Theorem 5.** On any uncountable Polish space \( X \), there is an \( S_\alpha \)-function \( f \) and there is an analytic subset \( C \) of the real line such that \( f^{-1}(C) \not\subset B_\alpha \).

**Proof.** It is known that \( B_\alpha \) is not closed under operation \( A \) (cf. [2]). Let \( \{ Z_{n_1} \ldots n_k \} \subset B_\alpha \) be such that \( \bigcap_{k=1}^{\infty} Z_{n_1} \ldots n_k \not\subset B_\alpha \), where \( \mathcal{R} \) denotes the family of all sequences of positive integers and \( n = (n_1, n_2, \ldots) \). We can find countably many sets \( \{ A_i \} \) in \( S_\alpha \) such that for all \( n \) and \( k \), \( Z_{n_1} \ldots n_k \in \sigma(A_i) \). Let \( f(x) = \sum_{i=1}^{\infty} (2/3^i) I_{A_i}(x) \). As the sum of two \( S_\alpha \)-functions, a positive constant multiple of an \( S_\alpha \)-function and the limit of an increasing sequence of \( S_\alpha \)-functions are all \( S_\alpha \)-functions, \( f \) is an \( S_\alpha \)-function. As \( f^{-1}(\mathcal{B}) = \sigma(A_i) \) where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \( R \), we can find, for all \( n \) and \( k \), \( B_{n_1} \ldots n_k \subset \mathcal{B} \) such that \( f^{-1}(B_{n_1} \ldots n_k) = Z_{n_1} \ldots n_k \). Let \( C = \bigcup_{k=1}^{\infty} B_{n_1} \ldots n_k \). Then \( C \) is analytic and \( f^{-1}(C) = \bigcup_{k=1}^{\infty} Z_{n_1} \ldots n_k \not\subset B_\alpha \).

**Remark.** Let \( X \) be any set and \( L \) a \( \sigma \)-additive lattice on \( X \) containing \( X \) and the null set, such that \( \sigma(L) \) is not closed under operation \( A \). We call a real valued function \( f \) on \( X \) an \( L^* \)-function if for every \( c \), \( \{ x : f(x) > c \} \subset L \). Evidently \( f^{-1}(B) \subset \sigma(L) \). However, we can find an analytic set \( C \) and an \( L^* \)-function \( f \) such that \( f^{-1}(C) \not\subset \sigma(L) \). The proof is similar to that of Theorem 5.

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Statistics-Mathematics Division, Indian Statistical Institute, Calcutta, India