A NOTE ON SOME PROPERTIES OF \(\mathcal{A}\)-FUNCTIONS

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Abstract. This note deals with \((\mathcal{M}, \ast)\) functions for various families \(\mathcal{M}\). It is shown that if \(\mathcal{M}\) is the family of Borel sets of additive class \(\alpha\) on a metric space \(X\), then \((\mathcal{M}, \ast)\) functions are just the functions of the form \(\sup_y g(x, y)\) where \(g: X \times R \to R\) is continuous in \(y\) and of class \(\alpha\) in \(x\). If \(\mathcal{M}\) is the class of analytic sets in a Polish space \(X\), then the \((\mathcal{M}, \ast)\) functions dominating a Borel function are just the functions \(\sup_y g(x, y)\) where \(g\) is a real valued Borel function on \(X^2\). It is also shown that there is an \(\mathcal{A}\)-function \(f\) defined on an uncountable Polish space \(X\) and an analytic subset \(C\) of the real line such that \(f^{-1}(C) \not\in \sigma\)-algebra generated by the analytic sets on \(X\).

1. Introduction. Let \(X\) be any set and \(\mathcal{M}, \mathcal{N}\) be classes of subsets of \(X\). Following Hausdorff, we call a real valued function \(f\) on \(X\) a function of class \((\mathcal{M}, \ast)\) if \(\{x: f(x) > c\}\) is in \(\mathcal{M}\) for every \(c\). If \(\{x: f(x) > c\}\) is in \(\mathcal{N}\) for every \(c\), \(f\) is said to be of class \((\ast, \mathcal{N})\). Set \((\mathcal{M}, \mathcal{N}) = (\mathcal{M}, \ast) \cap (\ast, \mathcal{N})\).

If \(X\) is a metric space and \(\mathcal{M}\) is the family of sets of additive Borel class \(\alpha\), then functions of class \((\mathcal{M}, \ast)\) are called \(\alpha\)-functions; if \(X\) is Polish and \(\mathcal{M}\) is the family of analytic sets, they are called \(\mathcal{A}\)-functions. We shall prove the following theorems:

**Theorem 1.** Let \(f\) be a real valued function on a metric space \(X\). Then \(f\) is an \(\alpha\)-function if, and only if, there is a real valued function \(g\) defined on \(X \times R\), where \(R\) is the real line, such that \(g(x, y)\) is a continuous function of \(y\) for fixed \(x\), is of class \(\alpha\) in \(x\) for fixed \(y\) and \(f(x) = \sup_y g(x, y)\).

**Theorem 2.** Let \(X\) be a Polish space and let \(f\) be a real valued function on \(X\) which is bounded below. Then \(f\) is an \(\mathcal{A}\)-function if, and only if, there is a real valued Borel function \(g\) on \(X^2\) such that \(f(x) = \sup_y g(x, y)\).

**Theorem 3.** Let \(\mathcal{A}\) be the \(\sigma\)-algebra generated by analytic sets on an uncountable Polish space \(X\). There is an \(\mathcal{A}\)-function \(f\) on \(X\) and an analytic subset \(C\) of the real line such that \(f^{-1}(C) \not\in \mathcal{A}\).

Theorem 3 answers in the negative a question raised by David Blackwell.

2. Proof of Theorem 1. We define a complete ordinary function system on a set \(X\) as a system \(F\) of real valued functions on \(X\) satisfying:
(a) Every constant function is in \( F \).
(b) If \( f, g \in F \), then \( \max(f, g), \min(f, g), f \pm g, f \cdot g \in F \). If \( g \) does not vanish anywhere, then \( f/g \in F \).
(c) If \( f_n \in F \) for all \( n \) and \( f_n \) converges uniformly to \( f \), then \( f \in F \).

We first prove the following:

**Theorem 4.** Let \( F \) be a complete ordinary function system on a set \( X \). Let \( P, Q \) be the families of sets \( \{x: h(x) > c\}, \{x: h(x) \geq c\} \), for \( h \in F \) and \( c \) real, respectively. \( f \in (P, \bullet) \) if, and only if, there is a real valued function \( g \) defined on \( X \times R \) such that \( g(x, y) \)

(a) is continuous in \( y \) for fixed \( x \),
(b) is in \( F \) for fixed \( y \), and
(c) \( \sup_y g(x, y) = f(x) \).

**Proof.** Suppose \( g(x, y) \) is a function on \( X \times R \) satisfying conditions (a) and (b) and suppose \( \sup_y g(x, y) \) exists and is \( f(x) \). Let \( c \) be any real number. Then

\[
    f(x) > c \iff \exists y \{g(x, y) > c\} \iff \exists y \{y \text{ is rational and } g(x, y) > c\},
\]

since \( g(x, y) \) is continuous in \( y \). Thus

\[
    \{x: f(x) > c\} = \bigcup_{y \text{ rational}} \{x: g(x, y) > c\}.
\]

For fixed \( r \), \( g(x, r) \in F \) and hence \( \{x: g(x, r) > c\} \in P \). Now as \( P \) is closed under countable unions (cf. [1]), \( \{x: f(x) > c\} \in P \).

Conversely, suppose \( f \in (P, \bullet) \). It is shown in [1] that there is an increasing sequence \( \{f_n\} \) in \( F \) which converges to \( f \). Define \( g \) on \( X \times R \) by \( g(x, y) = (f_{n+1}(x) - f_n(x))(|y| - n) + f_n(x) \) for \( |y| \in [n, n + 1] \). It is easy to see that \( g \) is well defined for all \( (x, y) \) and satisfies (a) and (b). As \( f_n(x) \leq g(x, y) \leq f_{n+1}(x) \) for \( |y| \in [n, n + 1] \) and \( \sup_y f_n(x) = f(x) \), \( \sup_y g(x, y) = f(x) \).

Theorem 1 follows from Theorem 4 and the following:

**Lemma.** Let \( F \) be the family of all functions of class \( \alpha \) on a Polish space \( X \). Then \( F \) is a complete ordinary function system and the sets of the form \( \{x: f(x) > c\}, f \in F, c \) real, are just the sets of additive Borel class \( \alpha \).

**Proof.** It is shown in [3] that \( F \) forms a complete ordinary function system.

Any set of the form \( \{x: f(x) > c\}, f \in F, c \) real, is clearly of additive Borel class \( \alpha \). Let \( A \) be any set of additive Borel class \( \alpha \). If \( \alpha = 0 \), \( A \) is a cozero set and hence \( A = \{x: f(x) > 0\} \) for some continuous function \( f \). If \( \alpha > 0 \), then we can write \( A = \bigcup_{n=1}^{\infty} A_n \) where the \( A_n \)'s are ambiguous of class \( \alpha \). Let \( f(x) = \sum_{n=1}^{\infty} 2^{-n} I_{A_n}(x) \) where \( I_{A_n} \) denotes the indicator function of \( A_n \). As \( I_{A_n} \) is of class \( \alpha \), \( f \) is of class \( \alpha \) and \( A = \{x: f(x) > 0\} \).

3. **Proof of Theorem 2.** If \( f(x) = \sup_y g(x, y) \) where \( g \) is Borel measurable, it is shown in [3] that \( f \) is an \( A \)-function. For this, \( f \) need not be bounded below.

Let \( f \) be an \( A \)-function on \( X \) such that \( f(x) \geq a \) for a fixed real number \( a \). Without loss of generality, we take \( X = R \). Let \( (r_n) \) enumerate all rationals. Let \( A = \{(x, y): f(x) > y\} \). Then \( A = \bigcup_n \{(x, y): f(x) > r_n > y\} \) and hence is analytic. Let \( B \subset R^3 \) be a Borel set such that \( A = \) projection of \( B \) i.e. \( (x, y) \in A \iff \exists z((x, y, z) \in B) \). Let \( k: R^3 \rightarrow R^3 \) be defined by

\[
    k(x, y, z) = \begin{cases} (x, y, z) & \text{if } (x, y, z) \in B, \\ (a, a, a) & \text{otherwise.} \end{cases}
\]
Then, as $k$ is Borel measurable so is $\tau_2 k$ where $\tau_2$ denotes projection to the second coordinate and

$$
\tau_2 k(x,y,z) = \begin{cases} y & \text{if } (x,y,z) \in B, \\
 a & \text{otherwise}. 
\end{cases}
$$

Thus $\sup_{(y,z)} \tau_2 k(x,y,z) = \sup_{(y,z)} \{ \{ y : y < f(x) \} \cup \{ a \} \} = f(x)$. Let $\phi$ be a Borel isomorphism from $R$ onto $R^2$. Let $h : R^2 \to R^3$ be defined by $h(x,y) = (x,\phi(y))$ and let $g(x,y) = \tau_2 kh(x,y)$. Then $g$ is Borel measurable and $f(x) = \sup_y \tau_2 k(x,\phi(y)) = \sup_y g(x,y)$.

**Remark.** It is easy to see that Theorem 2 holds even if the condition "$f$ is bounded below" is replaced by "$f$ dominates a Borel function". Thus an $A$-function is of the form $\sup_y g(x,y)$ for some Borel measurable $g$ if, and only if, it dominates a Borel function. Equivalently, every $A$-function is of the form $\sup_y g(x,y)$ for some Borel measurable $g$ if, and only if, given an ascending sequence of analytic sets $\{ A_n \}$ such that $\bigcup_{n=1}^{\infty} A_n = X$, there is an ascending sequence $\{ B_n \}$ of Borel sets such that $B_n \subset A_n$ and $\bigcup_{n=1}^{\infty} B_n = X$. However, we do not know if this condition always holds.

4. **Proof of Theorem 3.** In $X$, we put $S_0 = \text{the family of open sets}$, $B_0 = \sigma(S_0)$ and, for $0 < \alpha < \omega_1$, $S_\alpha = \sigma(\cup_{\beta < \alpha} S_\beta)$ and $B_\alpha = \sigma(S_\alpha)$ where, for any family of sets $G$, $\sigma(G)$ denotes the $\sigma$-algebra generated by $G$ and $\sigma(G)$ denotes the smallest family containing $G$ and closed under operation $A$.

We call $(S_\alpha, *)$ functions $S_\alpha$-functions. Theorem 3 is obtained from the following more general theorem by putting $\alpha = 1$.

**Theorem 5.** On any uncountable Polish space $X$, there is an $S_\alpha$-function $f$ and there is an analytic subset $C$ of the real line such that $f^{-1}(C) \notin B_\alpha$.

**Proof.** It is known that $B_\alpha$ is not closed under operation $A$ (cf. [2]). Let $(Z_{n_1}, \ldots, Z_{n_k}) \subset B_\alpha$ be such that $\bigcap_{n \in \mathbb{N}} \bigcap_{k=1}^{\infty} Z_{n_1} \cdots Z_{n_k} \notin B_\alpha$, where $\mathbb{N}$ denotes the family of all sequences of positive integers and $n = (n_1, n_2, \ldots)$. We can find countably many sets $\{ A_i \}$ in $S_\alpha$ such that for all $n$ and $k$, $Z_{n_1} \cdots Z_{n_k} \in \sigma(\{ A_i \})$.

Let $f(x) = \sum_{i=1}^{\infty} (2/3^i) I_{A_i}(x)$. As the sum of two $S_\alpha$-functions, a positive constant multiple of an $S_\alpha$-function and the limit of an increasing sequence of $S_\alpha$-functions are all $S_\alpha$-functions, $f$ is an $S_\alpha$-function. As $f^{-1}(B) = \sigma(\{ A_i \})$ where $B$ is the Borel $\sigma$-algebra on $R$, we can find, for all $n$ and $k$, $B_{n_1} \cdots B_{n_k} \in B$ such that $f^{-1}(B_{n_1} \cdots B_{n_k}) = Z_{n_1} \cdots Z_{n_k}$. Let $C = \bigcup_{n \in \mathbb{N}} \cap_{k=1}^{\infty} B_{n_1} \cdots B_{n_k}$. Then $C$ is analytic and $f^{-1}(C) = \bigcup_{n \in \mathbb{N}} \cap_{k=1}^{\infty} Z_{n_1} \cdots Z_{n_k} \notin B_\alpha$.

**Remark.** Let $X$ be any set and $L$ a $\sigma$-additive lattice on $X$ containing $X$ and the null set, such that $\sigma(L)$ is not closed under operation $A$. We call a real valued function $f$ on $X$ an $L^*$-function if for every $c$, $\{ x : f(x) > c \} \in L$. Evidently $f^{-1}(B) \subset \sigma(L)$. However, we can find an analytic set $C$ and an $L^*$-function $f$ such that $f^{-1}(C) \notin \sigma(L)$. The proof is similar to that of Theorem 5.

**Acknowledgement.** I am grateful to Dr. Ashok Maitra for his many suggestions. I am indebted to Dr. B. V. Rao for various discussions and for greatly simplifying and improving the proof of Theorem 3. I am also grateful to Dr. M. G. Nadkarni and Dr. K. P. S. B. Rao for some discussions. I am grateful to the referee for his suggestions.
REFERENCES


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