ON hom dim $MU_*(X \times Y)$

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Abstract. Let $p$ be a prime and $B\mathbb{Z}/p$ the classifying space for the cyclic group $\mathbb{Z}/p$ of prime order $p$. A finite complex $X$ is constructed such that

$$\text{hom} \cdot \dim_{MU_*} MU_*(X \times B\mathbb{Z}/p) > \text{hom} \cdot \dim_{MU_*} MU_*(X) + \text{hom} \cdot \dim_{MU_*} MU_*(B\mathbb{Z}/p).$$

It has been widely expected that

$$\text{hom} \cdot \dim_{MU_*} MU_*(X \times Y) \leq \text{hom} \cdot \dim_{MU_*} MU_*(X) + \text{hom} \cdot \dim_{MU_*} MU_*(Y)$$

for $X$ and $Y$ CW complexes of finite type and $MU_*(\cdot)$ the complex bordism homology functor [2], [5, (6)]. Of particular interest has been the case $X = B\mathbb{Z}/p = Y$, where $B\mathbb{Z}/p$ is the classifying space of the cyclic group $\mathbb{Z}/p$ of prime order $p$, as in this case the inequality would imply an affirmative solution to a conjecture of Conner and Floyd [1, pp. 130–131]. The following is therefore something of a surprise.

Theorem. For each prime $p$ there is a finite CW complex $X$ with

$$\text{hom} \cdot \dim_{MU_*} MU_*(X) = 1$$

such that

$$\text{hom} \cdot \dim_{MU_*} MU_*(X \times B\mathbb{Z}/p) \geq 3 > 1 + 1$$

$$= \text{hom} \cdot \dim_{MU_*} MU_*(X) + \text{hom} \cdot \dim_{MU_*} MU_*(B\mathbb{Z}/p).$$

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To construct the relevant complex $X$ we consider (for suitably large $n$) the pushout diagram

$$
\begin{array}{ccc}
M(p; n + 2(p - 1)) & \xrightarrow{p^{p+1}} & M(p^{p+2}; n + 2(p - 1)) \\
A \downarrow & & \downarrow \\
M(p, n) & \xrightarrow{\text{inclusion}} & X
\end{array}
$$

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where \( M(t; m) = S^m \cup_i e^{m+1}, A \) is the map called \( \alpha \) in [5] and \( \Phi \) in [3], and \( p_{p+1} \) is the map of degree \( p^{p+1} \) on the bottom cell. There is thus a cofibration

\[
M(p; n + 2(p - 1)) \xrightarrow{A_{*} - p_{p+1}} M(p, n) \vee M(p^{p+2}, n + 2(p - 1)) \to X
\]

giving an exact triangle

\[
MU_{*}(M(p, n + 2(p - 1))) \xrightarrow{j_{*}} MU_{*}(M(p, n)) \oplus MU_{*}(M(p^{p+2}, n + 2(p - 1))) \xrightarrow{\partial_{*}} MU_{*}(X)
\]

A moment's reflection shows that the horizontal map is monic, whence \( \partial_{*} = 0 \), and the resulting short exact sequence implies

**Proposition.** With the preceding notations, \( MU_{*}(X) \) is generated by two classes, \( u \in MU_{n}(X), w \in MU_{n+2(p-1)}(X) \), satisfying the relations \( pu = 0, [CP(p - 1)]u = p_{p+1}w. \) Moreover \( \text{hom dim}_{MU_{*}}MU_{*}(X) = 1. \)

**Proof.** All that remains to be proved is the assertion about projective dimension. To this end note there is a commutative diagram

\[
MU_{*}(M(p; n)) \oplus MU_{*}(M(p^{p+2}; n + 2(p - 1))) \to MU_{*}(X) \to 0
\]

\[
\begin{array}{ccc}
H_{*}(M(p; n); \mathbb{Z}) \oplus H_{*}(M(p^{p+2}; n + 2(p - 1)); \mathbb{Z}) & \xrightarrow{j_{*}} & H_{*}(X; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_{*}(M(p, n + 2(p - 1))) & \xrightarrow{A_{*} - p_{p+1}} & H_{*}(M(p, n + 2(p - 1)))
\end{array}
\]

Since \( A_{*} = 0 \) and \( p_{p+1} \) is monic, it follows that \( j_{*} \) is epic, whence the commutative square shows the Thom map \( \mu: MU_{*}(X) \to H_{*}(X; \mathbb{Z}) \) is epic and the result follows from [2, 3.11]. \( \square \)

**Proof of Theorem.** Recall [1, 46.3] that \( MU_{*}(B\mathbb{Z}/p) \) is generated by classes \( \alpha_{2k-1} \in MU_{2k-1}(B\mathbb{Z}/p) \) of additive order \( p_{p+1} \) where \( 2\alpha(p - 1) < 2k - 1 < 2(a + 1)(p - 1) [1, 36.1]. \) There is (among many others!) the relation [1, p. 145(*)]

\[
[V^{2p^2-2}]\alpha_1 + [CP(p - 1)]\alpha_{2p(p - 1)+1} \in pMU_{*}(B\mathbb{Z}/p),
\]

where \( [V^{2p^2-2}] \) is a Milnor manifold of dimension \( 2p^2 - 2. \) So write

\[
[V^{2p^2-2}]\alpha_1 = px - [CP(p - 1)]\alpha_{2p(p - 1)+1}.
\]

From the Künneth exact sequence [2, 8.4] we see that

\[
u \otimes \alpha_1 \neq 0 \in MU_{*}(X \times B\mathbb{Z}/p).
\]
Note

\[ [V^{2p^2-2}]u \otimes \alpha_1 = u \otimes [V^{2p^2-2}]\alpha_1 \]
\[ = u \otimes (px - [CP(p - 1)]\alpha_{2p(p-1)+1}) \]
\[ = pu \otimes x - [CP(p - 1)]u \otimes \alpha_{2p(p-1)+1} \]
\[ = 0 - p^{p+1}w \otimes \alpha_{2p(p-1)+1} \]
\[ = w \otimes p^{p+1}\alpha_{2p(p-1)+1} = u \otimes 0 = 0. \]

Therefore the annihilator ideal \( A(u \otimes \alpha_1) \) contains \([V^{2p^2-2}]\). From degree considerations, \( u \otimes \alpha_1 \) is primitive; so, by the Ballantine lemma [3, II, 2.1], it follows that \( A(u \otimes \alpha_1) \) also contains \([CP(p - 1)]\) and \( p \). Hence,

\[ \text{hom } \cdot \text{dim}_{MU_*}MU_*(X \times B\mathbb{Z}/p) \geq 3 \]

by [3, 5.3].

\textbf{Remark.} By replacing \( B\mathbb{Z}/p \) by a suitable large lens space \( L(2m - 1; p) \), we obtain finite complexes \( X, Y \) with \( MU \) \text{hom } \cdot \text{dim} 1, whose Cartesian product has \( MU \) \text{hom } \cdot \text{dim} at least 3.

\textbf{References}


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