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CORRIGENDUM AND ADDENDUM TO
"PSEUDO-MATCHINGS OF A BIPARTITE GRAPH"

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Abstract. For an undirected bipartite graph $G$ conditions on $\deg x + \deg y$ for $xy \notin G$ ensure sets of independent lines extend to hamiltonian cycles.

Let $X, Y$ be two finite sets each with $n > 2$ elements. Also let $G$ be an undirected bipartite graph with points $X \cup Y$ and lines $xy$ with $x \in X, y \in Y$. We will assume $\deg x + \deg y > \delta$ whenever $xy \notin G$. By HC and HP we mean hamiltonian cycle and path respectively. Consider the following:

**Theorem 1.** Every set of independent lines of $G$ extends to a matching iff each connected component of $G$ is a complete graph $K_{m,m}$.

**Theorem 2.** If $\deg v > \frac{1}{2}n$ for each point $v$ then $G$ has an HC.

**Theorem 3.** (i) If $\delta = 1 + m$ where $1 < l < m < n$ then every set of $\leq l$ independent lines of $G$ extends to a set of $m$ independent lines.

(ii) If $\delta = n + 1$ every set of independent lines of $G$ extends to a matching and every matching extends to an HP. Also some $n - 1$ lines of any matching extend to an HC.

(iii) If $\delta = n + 2$ every matching of $G$ extends to an HC.

(iv) If $\delta = \frac{1}{2}(4n + 1)$ any set of independent lines of $G$ extends to an HC.

(v) If $\delta = n + 1 + \frac{1}{2}k$ then any path of length $k$ in $G$ extends to an HC.

The proof of Theorem 1 is easy. Theorem 2 appeared in [3] but this writer introduced an error into the proof by claiming that $n + 2$ can be replaced by $n + 1$ in Theorem 3(iii). Once this is realised the necessary corrections for [3] are readily made. In fact a stronger form of Theorem 2 appeared in [6]. Clearly Theorem 3(ii) is stronger than Theorem 2. That $G$ has an HC when $\delta = n + 1$ was proved in [2]. It is interesting to compare the results here with those in [4]. Theorem 3(i) follows from the result about the largest set of independent lines in $G$ (see [1, Theorem 4, p. 98]). We will need a result [7, Theorem 2D] of D. R. Woodall, whose kind suggestions improved an earlier version of this note, namely

**Theorem 4.** Let $F$ be a directed graph on $n$ vertices with out deg $u + \text{in deg } v \geq \Delta$ whenever $u \rightarrow v$ is not an arc of $F$.

(i) If $\Delta = n - 1$ then $F$ has a directed HP.

(ii) If $\Delta = n$ then $F$ has a directed HC.

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Notice that Theorem 4(i) follows from Theorem 4(ii) by adding a new point \( w \) to \( F \) and all possible arcs between \( w \) and \( F \). In the same way Meyniel's Theorem [5] shows that if

\[
\text{out deg } u + \text{ in deg } u + \text{out deg } v + \text{ in deg } v \geq 2n - 3
\]

whenever \( u \to v \) and \( v \to u \) are not arcs of \( F \) then \( F \) has a directed HP, but we will not need this stronger result.

**Proof of Theorem 3(ii) and 3(iii).** By Theorem 3(i) there is a matching; let \( X, Y \) be numbered so it is \( x_i y_i \) for \( 1 \leq i \leq n \). Draw a directed graph \( F \) with vertices \( z_1, \ldots, z_n \) in which \( z_i \to z_j \) if \( i \neq j \) and \( x_i y_j \in G \). Then \( F \) has \( \Delta = \delta - 2 \). By Theorem 4 there is an HP or an HC in \( F \) which yields an HP or an HC as the case may be of \( G \) containing the given matching. For the last part of Theorem 3(ii) we suppose the HP is \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \). We suppose \( x_1 y_n \notin G \) for otherwise we have the desired HC. Then \( \text{deg } x_1 + \text{deg } y_n \geq n + 1 \) and there is an \( i \) in \( 1 < i < n \) with \( x_i y_i \) and \( x_i y_n \) both in \( G \), so

\[
x_1, y_1, x_2, y_2, \ldots, x_i, y_i, x_n, y_n, x_{n-1}, y_{n-1}, \ldots, y_{i+1}, x_{i+1}, y_i, x_1
\]

is the desired HC. To see that 3(iii) is best possible consider two complete bipartite graphs with exactly one edge in common and take any matching which contains this edge.

**Proof of Theorem 3(iv).** Suppose \( n \geq 3 \) and there is a graph \( G \), with the maximum possible number of lines, such that some set \( L \) of independent lines does not extend to an HC. Then there is a line \( xy \) not in \( G \) and \( L \) extends to an HC in \( G + xy \). Let the HC be

\[
x = x_1, y_1, x_2, y_2, \ldots, x_n, y_n = y, x_1.
\]

Of course adjacent lines of this cycle are not both in \( L \). Let \( I \) be the set of integers \( i \) such that \( xy_i \) and \( x_i y \) are both in \( G \). Then \( 1, n \notin I \) and if \( k = |I| \) we have \( \frac{1}{2}(4n + 1) \leq \text{deg } x + \text{deg } y \leq n + k \) so \( k \geq 2 \). Also \( x_i y_i \in L \) for \( i \in I \) as otherwise

\[
x_1, y_1, x_2, y_2, \ldots, x_{i-1}, y_{i-1}, x_i, y_i, x_n, y_n, x_{n-1}, y_{n-1}, \ldots, y_{i+1}, x_{i+1}, y_i, x_1
\]

is an HC containing \( L \). Next notice that if \( s \) is the smallest integer in \( I \) then \( y_{s-1} x_{i+1} \notin G \) for any given \( i \in I \), for otherwise

\[
x_1, y_1, x_2, y_2, \ldots, x_{s-1}, y_{s-1}, x_{s-1}, y_{s-1}, x_{s+1}, y_{s+1}, x_{i+2}, y_{i+2}, \ldots, x_n, y_n, x_{s-1}, y_{s-1}, \ldots, y_i, x_i, y_i, x_1
\]

is an HC containing \( L \). It follows that \( \text{deg } y_{s-1} \leq n - k \) and by symmetry \( \text{deg } x_{r+1} \leq n - k \) where \( r \) is the largest integer in \( I \). Hence \( 2(n - k) \geq \text{deg } y_{s-1} + \text{deg } x_{r+1} \geq \frac{1}{2}(4n + 1) \) which contradicts the earlier result \( \frac{1}{2}(4n + 1) \leq n + k \).

Let \( n = 3m + \lambda \) where \( \lambda \in \{-1,0,1\} \), and \( W \subset X \) with \( |W| = m \), and \( Z \subset Y \) with \( |Z| = 2m - 1 \). Let \( G \) be the graph containing all lines except \( xy \) with \( x \in W \) and \( y \in Z \). Then any matching of \( Z \) into \( X \setminus W \) does not extend to an HC. This example has \( \text{deg } x + \text{deg } y = \left[ 1/2(4n + 1) \right] - 1 \) and so shows
Theorem 3(iv) is best possible when $\lambda = -1$. Probably $\frac{1}{2}(4n + 1)$ can be replaced by its integral part for $n \geq 5$, but it seems to be hard to obtain this slight improvement.

Notice that if $\delta = n + |L| \geq n + 2$ then $L$ extends by (i) to a matching, which in turn extends to an HC by (iii), but is this result best possible?

Proof of Theorem 3(v). Suppose there is a graph $G$, with the maximum possible number of lines, such that some path $P$ of even length $k$ does not extend to an HC. Then there is a line $xy$ not in $G$ and $P$ extends to an HC in $G + xy$. The value of $\delta$ ensures we can choose an edge $x'y'$ in this HC but not in $P$, such that $xy'$, $x'y$ both lie in $G$. Hence we get a contradictory HC containing $P$. Examples with $G$ consisting of two overlapping complete bipartite graphs and $P$ filling the intersection, show the result best possible for $k \leq 2n - 4$. For $k \geq 2n - 3$ by inspection the graph must be complete if $P$ is to extend to an HC.

References


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