A REMARK ON THE RESTRICTION MAP IN FIELD FORMATION

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Abstract. In this note we point out that in a field formation \((G, \{G_F\}, A)\), if \(h_2(K/F) = [K: F]^c\) for every normal layer \(K/F\) with a fixed integer \(c \geq 0\), then for every tower \(F \subset E \subset K\) with \(K/F\) normal, the restriction map \(H^2(K/F) \to H^2(K/E)\) is surjective, and give an example with \(c = 2\).

Let \((G, \{G_F\}, A)\) be a field formation. (For the notations and basic facts, see [1, Chapter 14, pp. 197-209].) Then for every tower \(F \subset K \subset L\) with \(K/F\) and \(L/F\) normal, the sequence

\[
0 \to H^2(K/F) \to H^2(L/F) \to H^2(L/K) \to 0
\]

is exact. We shall prove that if \(h_2(K/F) = [K: F]^c\) for every normal layer \(K/F\) with a fixed integer \(c \geq 0\), then for every tower \(F \subset E \subset K\) with \(K/F\) normal, \(\text{res}: H^2(K/F) \to H^2(K/E)\) is surjective.

If \(F \subset K \subset L\) with \(K/F\) and \(L/F\) normal, then the exact sequence (1) gives that the order of the image \(\text{res} H^2(L/F)\) is equal to \([L: F]^c/[K: F]^c = [L: K]^c = h_2(L/K)\). Thus \(\text{res}: H^2(L/F) \to H^2(L/K)\) is surjective in this case.

Let \(F \subset E \subset K\) with \(K/F\) normal. For each prime \(p\), the restriction map takes the \(p\)-primary component \(H^2(K/F)_p\) of \(H^2(K/F)\) into the \(p\)-primary component \(H^2(K/E)_p\) of \(H^2(K/E)\). Thus it is enough to show that \(\text{res}: H^2(K/F)_p \to H^2(K/E)_p\) is surjective for every \(p\).

Let \(G_{K/E_0}\) be a \(p\)-Sylow subgroup of \(G_{K/E}\). Then by a Sylow theorem, there exists a chain

\[
G_{K/E_0} = G_{K/F_0} \subset G_{K/F_1} \subset \cdots \subset G_{K/F_r}
\]

of \(p\)-subgroups of \(G_{K/F}\) such that \(G_{K/F_i}\) is normal in \(G_{K/F_{i-1}}\) for each \(i = 1, \ldots, r\) and \(G_{K/F_r}\) is a \(p\)-Sylow subgroup of \(G_{K/F}\). We know that the restriction map takes \(H^2(K/E)_p\) injectively into \(H^2(K/E_0)\). In our case,

\[
\text{res}: H^2(K/E)_p \to H^2(K/E_0)
\]

is bijective because both \(H^2(K/E)_p\) and \(H^2(K/E_0)\) have the same order \([K: E_0]^c\). Likewise

\[
\text{res}: H^2(K/F)_p \to H^2(K/F_0)
\]

is bijective. Thus it is sufficient to show that
is surjective. This map factors as
\[ H^2(K/F_0) \xrightarrow{\text{res}} H^2(K/F_1) \xrightarrow{\text{res}} \cdots \xrightarrow{\text{res}} H^2(K/F_i). \]
Since \( K/F_i \) and \( F_i/F_{i-1} \) are normal, each factor
\[ \text{res}: H^2(K/F_{i-1}) \to H^2(K/F_i) \]
is surjective. Thus the composite (2) of these is surjective. This completes the proof of the remark.

We now offer an example of field formation in which \( h_2(K/F) = [K: F]^2 \) for every normal layer \( K/F \). Let \( p \) be a rational prime and \( Q_p \) be the rational \( p \)-adic number field. Let \( P = Q_p(t) \), the field of formal power series in \( t \) over \( Q_p \). Let \( \Omega \) be the splitting field of the polynomials \( X^n - t \) over \( P \) for all integers \( n > 0 \) not divisible by \( p \). Given a finite extension \( F \) of \( P \) in \( \Omega \), let \( G_F \) be the Galois group of \( \Omega/F \). Then \( (G_F, \Omega, \Omega^\times) \), where \( G = G_p \) and \( \Omega^\times \) is the multiplicative group of \( \Omega \), is a field formation. We claim that \( h_2(K/F) = [K: F]^2 \) for every normal layer \( K/F \) in \( \Omega/P \).

The ground field \( P \) is complete under the nonarchimedian valuation \( | \cdot | \) given by \( |x| = e^{-r} \) if
\[ x = \sum_{k \geq r} a_k t^k, \quad a_k \in Q_p, \quad a_r \neq 0, \]
and the valuation is extended to \( \Omega \). Given a field \( K \) in the formation, let \( \Theta_K \), \( M_K \) and \( U_K \) be the valuation ring, its maximal ideal and the group of units in \( \Theta_K \), respectively. Let \( \bar{K} = \Theta_K/M_K \) and \( U_K^1 = 1 + M_K. \) The residue field \( \bar{K} \) is an abelian extension of \( Q_p \).

Since every normal layer in our formation is solvable, by induction we see that it is sufficient to prove the equality \( h_2(K/F) = [K: F]^2 \) for every cyclic layer \( K/F \) of prime degree. For this it is sufficient to establish the equality in the following two cases: (a) when \( K/F \) is unramified and (b) when \( K/F \) is totally ramified.

For any normal layer \( K/F \), we have \( H^q(G_K/F, U_K^1) = 0 \) for all \( q \) because \( x \mapsto x^n \) is an automorphism of \( U_K^1. \) Thus the exact sequence
\[ 0 \to U_K^1 \to U_K \to \bar{K}^\times \to 0 \]
gives that
\[ H^q(G_K/F, U_K) = H^q(G_K/F, \bar{K}^\times) \]
for all \( q \). While the exact sequence
\[ 0 \to U_K \to K^\times \xrightarrow{\nu_K} \mathbb{Z} \to 0, \]
where \( \nu_K \) is the exponential valuation on \( K \), gives the long exact sequence
\[ 0 \to H^2(G_K/F, U_K) \to H^2(K/F) \to H^2(G_K/F, \mathbb{Z}) \xrightarrow{\delta} \cdots. \]
Note that

\[ H^2(G_{K/F}, \mathbb{Z}) = H^1(G_{K/F}, \mathbb{Q}/\mathbb{Z}) = G_{K/F}^{\ast}, \]

the character group of \( G_{K/F} \).

**Case (a).** \( K/F \) is unramified. Since \( G_{K/F} = G_{K/F}^{\ast} \), (3) gives that

\[ H^q(G_{K/F}, U_K) = H^q(K/F). \]

Since the exact sequence (4) splits in this case, we get the exact sequence

\[ 0 \rightarrow H^2(K/F) \rightarrow H^2(K/F) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0. \]

Since \( K/F \) is abelian, \( |G_{K/F}^{\ast}| = [K: F] \). While by the local class field theory, \( h_2(K/F) = [\mathbb{K}: F] = [K: F] \). Thus we get that \( h_2(K/F) = [K: F]^2 \).

**Case (b).** \( K/F \) is totally ramified. Then \( K = F \) and this field contains a primitive \( n \)th root of unity, where \( n = [K: F] \), and \( K/F \) is cyclic. Thus

\[ H^3(G_{K/F}, \mathbb{K}^{\times}) = H^1(G_{K/F}, \mathbb{F}^{\times}) \]

is of order \( n \). Since \( H^3(K/F) = H^1(K/F) = 0 \) and \( |G_{K/F}^{\ast}| = n \),

\[ \delta: G_{K/F}^{\ast} \rightarrow H^3(G_{K/F}, \mathbb{K}) \]

is an isomorphism. Thus (5) gives that

\[ H^2(K/F) = H^2(G_{K/F}, \mathbb{K}^{\times}) = H^0(G_{K/F}, \mathbb{F}^{\times}) = \mathbb{F}^{\times}/\mathbb{F}^{\times n}. \]

But since \( (n, p) = 1 \), \( [\mathbb{F}^{\times}: \mathbb{F}^{\times n}] = n^2 \). Thus \( h_2(K/F) = [K: F]^2 \).

**Reference**


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