A REMARK ON THE RESTRICTION MAP IN FIELD FORMATION

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Abstract. In this note we point out that in a field formation \((G, \{G_F\}, A)\), if \(h_2(K/F) = [K: F]^c\) for every normal layer \(K/F\) with a fixed integer \(c \geq 0\), then for every tower \(F \subset E \subset K\) with \(K/F\) normal, the restriction map \(H^2(K/F) \to H^2(K/E)\) is surjective, and give an example with \(c = 2\).

Let \((G, \{G_F\}, A)\) be a field formation. (For the notations and basic facts, see [1, Chapter 14, pp. 197-209].) Then for every tower \(F \subset K \subset L\) with \(K/F\) and \(L/F\) normal, the sequence

\[0 \to H^2(K/F) \to H^2(L/F) \to H^2(L/K)\]

is exact. We shall prove that if \(h_2(K/F) = [K: F]^c\) for every normal layer \(K/F\) with a fixed integer \(c \geq 0\), then for every tower \(F \subset E \subset K\) with \(K/F\) normal, \(\text{res}: H^2(K/F) \to H^2(K/E)\) is surjective.

If \(F \subset K \subset L\) with \(K/F\) and \(L/F\) normal, then the exact sequence (1) gives that the order of the image \(\text{res} H^2(L/F)\) is equal to \([L: F]^c/[K: F]^c = [L: K]^c = h_2(L/K)\). Thus \(\text{res}: H^2(L/F) \to H^2(L/K)\) is surjective in this case.

Let \(F \subset E \subset K\) with \(K/F\) normal. For each prime \(p\), the restriction map takes the \(p\)-primary component \(H^2(K/F)_p\) of \(H^2(K/F)\) into the \(p\)-primary component \(H^2(K/E)_p\) of \(H^2(K/E)\). Thus it is enough to show that \(\text{res}: H^2(K/F)_p \to H^2(K/E)_p\) is surjective for every \(p\).

Let \(G_{K/E_0}\) be a \(p\)-Sylow subgroup of \(G_{K/E}\). Then by a Sylow theorem, there exists a chain

\[G_{K/E_0} = G_{K/E} \subset G_{K/E_{i-1}} \subset \cdots \subset G_{K/F_0}\]

of \(p\)-subgroups of \(G_{K/F}\) such that \(G_{K/F_i}\) is normal in \(G_{K/F_{i-1}}\) for each \(i = 1, \ldots, r\) and \(G_{K/F_0}\) is a \(p\)-Sylow subgroup of \(G_{K/F}\). We know that the restriction map takes \(H^2(K/E)_p\) injectively into \(H^2(K/E_0)\). In our case,

\[\text{res}: H^2(K/E)_p \to H^2(K/E_0)\]

is bijective because both \(H^2(K/E)_p\) and \(H^2(K/E_0)\) have the same order \([K: E_0]^c\). Likewise

\[\text{res}: H^2(K/F)_p \to H^2(K/F_0)\]

is bijective. Thus it is sufficient to show that
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res: $H^2(K/F_0) \to H^2(K/E_0)$

is surjective. This map factors as

$$ H^2(K/F_0) \xrightarrow{\text{res}} H^2(K/F_1) \xrightarrow{\text{res}} \cdots \xrightarrow{\text{res}} H^2(K/F_r). $$

Since $K/F_i$ and $F_i/F_{i-1}$ are normal, each factor

res: $H^2(K/F_{i-1}) \to H^2(K/F_i)$

is surjective. Thus the composite (2) of these is surjective. This completes the proof of the remark.

We now offer an example of field formation in which $h_2(K/F) = [K: F]^2$ for every normal layer $K/F$. Let $p$ be a rational prime and $Q_p$ be the rational $p$-adic number field. Let $P = Q_p(t)$, the field of formal power series in $t$ over $Q_p$. Let $\Omega$ be the splitting field of the polynomials $X^n - t$ over $P$ for all integers $n > 0$ not divisible by $p$. Given a finite extension $F$ of $P$ in $\Omega$, let $G_F$ be the Galois group of $\Omega/F$. Then $(G, [G_F], \Omega^\times)$, where $G = G_P$ and $\Omega^\times$ is the multiplicative group of $\Omega$, is a field formation. We claim that $h_2(K/F) = [K: F]^2$ for every normal layer $K/F$ in $\Omega/P$.

The ground field $P$ is complete under the nonarchimedian valuation $|\cdot|$ given by $|x| = e^{-r}$ if

$$ x = \sum_{k \geq r} a_k t^k, \quad a_k \in Q_p, \quad a_r \neq 0, $$

and the valuation is extended to $\Omega$. Given a field $K$ in the formation, let $\mathfrak{o}_K$, $M_K$ and $U_K$ be the valuation ring, its maximal ideal and the group of units in $\mathfrak{o}_K$, respectively. Let $K = \mathfrak{o}_K/M_K$ and $U_K^1 = 1 + M_K$. The residue field $\overline{K}$ is an abelian extension of $Q_p$.

Since every normal layer in our formation is solvable, by induction we see that it is sufficient to prove the equality $h_2(K/F) = [K: F]^2$ for every cyclic layer $K/F$ of prime degree. For this it is sufficient to establish the equality in the following two cases: (a) when $K/F$ is unramified and (b) when $K/F$ is totally ramified.

For any normal layer $K/F$, we have $H^q(G_K/F, U_K^1) = 0$ for all $q$ because $x \mapsto x^n$ is an automorphism of $U_K^1$. Thus the exact sequence

$$ 0 \to U_K^1 \to U_K \to \overline{K}^\times \to 0 $$

gives that

$$ H^q(G_K/F, U_K) = H^q(G_K/F, \overline{K}^\times) $$

for all $q$. While the exact sequence

$$ 0 \to U_K \to \overline{K}^\times \xrightarrow{\nu_K} \mathbb{Z} \to 0, $$

where $\nu_K$ is the exponential valuation on $K$, gives the long exact sequence

$$ 0 \to H^2(G_K/F, U_K) \to H^2(K/F) \to H^2(G_K/F, \mathbb{Z}) \xrightarrow{\delta} \cdots. $$
Note that
\[ H^2(G_{K/F}, \mathbb{Z}) = H^1(G_{K/F}, \mathbb{Q}/\mathbb{Z}) = G_{K/F}^*, \]
the character group of $G_{K/F}$.

**Case (a).** $K/F$ is unramified. Since $G_{K/F} = G_{K/F}$, (3) gives that
\[ H^q(G_{K/F}, U_K) = H^q(K/F). \]

Since the exact sequence (4) splits in this case, we get the exact sequence
\[ 0 \to H^2(K/F) \to H^2(K/F) \to \mathbb{Q}/\mathbb{Z} \to 0. \]

Since $K/F$ is abelian, $|G_{K/F}^*| = [K : F]$. While by the local class field theory, $h_2(K/F) = [K : F] = [K : F]$. Thus we get that $h_2(K/F) = [K : F]^2$.

**Case (b).** $K/F$ is totally ramified. Then $K = F$ and this field contains a primitive $n$th root of unity, where $n = [K : F]$, and $K/F$ is cyclic. Thus
\[ H^3(G_{K/F}, K^\times) = H^1(G_{K/F}, F^\times) \]
is of order $n$. Since $H^3(K/F) = H^1(K/F) = 0$ and $|G_{K/F}^*| = n$,
\[ \delta: G_{K/F}^* \to H^3(G_{K/F}, K) \]
is an isomorphism. Thus (5) gives that
\[ H^2(K/F) = H^2(G_{K/F}, K^\times) = H^0(G_{K/F}, F^\times) = F^\times/F^\times n. \]
But since $(n, p) = 1$, $[F^\times : F^\times n] = n^2$. Thus $h_2(K/F) = [K : F]^2$.

**Reference**


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