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INTERPOLATION FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND A RELATED TRIGONOMETRIC MOMENT PROBLEM

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ABSTRACT. A classical theorem of Hausdorff-Young shows that when $1 < p < 2$, the system of equations $\hat{\varphi}(n) = c_n$ $(-\infty < n < \infty)$ admits a solution φ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$. Here, as usual, $\hat{\varphi}$ denotes the complex Fourier transform of φ and q is the conjugate exponent given by $p^{-1} + q^{-1} = 1$. The purpose of this note is to show that if a set $\{\lambda_n\}$ of real or complex numbers is "sufficiently close" to the integers, then the corresponding system $\hat{\varphi}(\lambda_n) = c_n$ is also solvable for φ whenever $\{c_n\} \in l^p$. The proof is accomplished by establishing a similar interpolation theorem for a related class of entire functions of exponential type.

1. **Introduction.** A classical theorem of Hausdorff-Young shows that when $1 < p < 2$, the system of equations $\hat{\varphi}(n) = c_n$ $(-\infty < n < \infty)$ admits a solution φ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$. Here, as usual, $\hat{\varphi}$ denotes the complex Fourier transform of φ and q is the conjugate exponent given by $p^{-1} + q^{-1} = 1$. In this note we show that if a set $\{\lambda_n\}$ of real or complex numbers is "sufficiently close" to the integers, then the corresponding system

$$\hat{\varphi}(\lambda_n) = c_n \qquad (-\infty < n < \infty) \tag{1}$$

admits a solution φ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$. Specifically, we have the following result.

THEOREM 1. *Let $1 < p < 2$ and let q be the conjugate exponent. There exists a constant $L > 0$ with the following property: If $|\lambda_n - n| \leqslant L$, then the system* (1) *admits a solution φ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$.*

We prove Theorem 1 by establishing a similar interpolation theorem for a related class of entire functions of exponential type.

2. **Interpolation in a related Banach space of entire functions.** We denote by E_τ^p $(p \geqslant 1)$ the Banach space of entire functions of exponential type τ for which

$$\|f\|_p = \left\{\int_{-\infty}^{\infty} |f(x)|^p \, dx\right\}^{1/p} < \infty.$$

A sequence $\{\lambda_n\}$ of distinct real or complex numbers is said to be an *interpolating sequence* for E_τ^p if $TE_\tau^p \supset l^p$, where T is given by $Tf = \{f(\lambda_n)\}$. (Such sequences were studied extensively in [5] for the special cases $p = 1, 2$

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and the limiting case $p = \infty$. For general properties of the spaces E_τ^p see [2].) It is well known that E_τ^p is closed under differentiation and that

$$\|f'\|_p \leqslant \tau\|f\|_p. \tag{2}$$

A simple application of the closed graph theorem shows that if $\{\lambda_n\}$ is an interpolating sequence, then the unit ball in l^p can be interpolated in a uniformly bounded way [5]; that is, there exists a constant M such that whenever $c \in l^p$, $\|c\| \leqslant 1$, there corresponds at least one function f in E_τ^p for which $Tf = c$ and $\|f\| \leqslant M$. As a consequence [5], if the imaginary part of λ_n is uniformly bounded, then the λ_n are of necessity *separated*, that is $\inf|\lambda_n - \lambda_m| > 0$ $(n \neq m)$. But then, it follows that for every function f in E_τ^p,

$$\left\{\sum_n |f(\lambda_n)|^p\right\}^{1/p} \leqslant A\|f\|_p, \tag{3}$$

where $A = A(p, \tau, \{\lambda_n\})$ is an absolute constant, independent of f [6]. Hence, in this case, T is in fact a bounded linear transformation *into* l^p.

If S: $E_\tau^p \to l^p$ is defined by $Sf = \{f(\mu_n)\}$, then we shall wish to conclude that $SE_\tau^p = l^p$ knowing that $TE_\tau^p = l^p$ and that S is "close" to T. For this purpose, we will need the following interesting result of Bade and Curtis [1].

LEMMA 1. *Let X and Y be Banach spaces and T: $X \to Y$ a bounded linear transformation. Suppose that there exist constants $M > 0$ and $0 < \epsilon < 1$ with the following property: For each y in the unit ball of Y, there exists an x in X with $\|x\| \leqslant M$ and $\|Tx - y\| \leqslant \epsilon$. Then T is onto.*

The proof of the following lemma is similar to that given in [3] for the case $p = 2$.

LEMMA 2. *Let $\{\lambda_n\}$ be a separated sequence of points lying in a strip parallel to the real axis, and suppose that $|\mu_n - \lambda_n| \leqslant L$. Then for every function f belonging to E_τ^p, we have the inequality*

$$\left\{\sum_n |f(\mu_n) - f(\lambda_n)|^p\right\}^{1/p} \leqslant A(e^{\tau L} - 1)\|f\|_p, \tag{4}$$

where A is the same constant appearing in (3).

PROOF. Using Taylor's theorem, we write

$$f(\mu_n) - f(\lambda_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{k!}(\mu_n - \lambda_n)^k.$$

Then, for any $\rho > 0$,

$$f(\mu_n) - f(\lambda_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{\rho^k (k!)^{1/p}} \cdot \frac{\rho^k(\mu_n - \lambda_n)^k}{(k!)^{1/q}},$$

and hence, by Hölder's inequality,

$$|f(\mu_n) - f(\lambda_n)| \leqslant \left\{\sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^p}{\rho^{kp} k!}\right\}^{1/p} \cdot \left\{\sum_{k=1}^{\infty} \frac{(\rho L)^{kq}}{k!}\right\}^{1/q}.$$

Now, since $f^{(k)} \in E_\tau^p$, it follows from (2) and (3) that

$$\sum_n |f^{(k)}(\lambda_n)|^p \leqslant A^p \tau^{kp} \|f\|_p^p.$$

Therefore, we conclude that

$$\left\{\sum_n |f(\mu_n) - f(\lambda_n)|^p\right\}^{1/p} \leqslant A\|f\|_p \left\{\sum_{k=1}^{\infty} \frac{\tau^{kp}}{\rho^{kp} k!}\right\}^{1/p} \left\{\sum_{k=1}^{\infty} \frac{(\rho L)^q}{k!}\right\}^{1/q}$$
$$= A\|f\|_p \{e^{\tau^p \rho^{-p}} - 1\}^{1/p} \{e^{\rho^q L^q} - 1\}^{1/q},$$

and the result follows by taking $\rho = \tau^{1/q} L^{-1/p}$.

The proof of Theorem 1 will follow easily from the following interpolation theorem for E_τ^p.

THEOREM 2. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $\{\lambda_n\}$ is an interpolating sequence for E_τ^p, then there exists a constant $L > 0$ such that $\{\mu_n\}$ is also an interpolating sequence for E_τ^p whenever $|\mu_n - \lambda_n| \leqslant L$.*

PROOF. Since $\{\lambda_n\}$ is interpolating for E_τ^p, the unit ball of l^p can be interpolated in a uniformly bounded way. Thus, there exists a constant M such that whenever $\sum |c_n|^p \leqslant 1$, there exists a function f in E_τ^p with $f(\lambda_n) = c_n$ and $\|f\|_p \leqslant M$.

Let us define a mapping T: $E_\tau^p \to l^p$ by $Tf = \{f(\mu_n)\}$. The inequality (4) shows that T is a bounded linear transformation into l^p. We show that T is in fact *onto* l^p. Let $c = \{c_n\}$ belong to the unit ball of l^p and choose f in E_τ^p such that $\|f\|_p \leqslant M$ and $f(\lambda_n) = c_n$. Then (4) becomes

$$\|Tf - c\| \leqslant AM(e^{\tau L} - 1), \tag{5}$$

and since L can be chosen small enough so that the right-hand side of (5) is less than 1, the conclusion follows from Lemma 1.

COROLLARY. *If $1 < p < 2$, then $\{\lambda_n\}$ is an interpolating sequence for E_π^p whenever $|\lambda_n - n| \leqslant L$ and L is sufficiently small.*

PROOF. In view of Theorem 2, it is enough to show that the integers are an interpolating sequence for E_π^p. Suppose that $\{c_n\} \in l^p$. By the Hausdorff-Young theorem, there exists a function φ in $L^q(-\pi, \pi)$ such that $\hat{\varphi}(n) = c_n$ $(-\infty < n < \infty)$. Since $\{\hat{\varphi}(n)\} \in l^p$ and $p > 1$, it follows that $\hat{\varphi}(x) \in L^p(-\infty, \infty)$ [4]. Thus, the function $\hat{\varphi}(z)$ belongs to E_π^p and the proof is complete.

REMARK. For $p = 1$, the integers fail to be an interpolating sequence for E_π^1 for the trivial reason that the nonzero integers are already a set of uniqueness. It was shown in [5], however, that $TE_\tau^1 = l^1$ for every $\tau > \pi$.

3. **Proof of Theorem 1.** The proof of Theorem 1 follows immediately from the above corollary since every function f belonging to E_π^p is of the form $f = \hat{\varphi}$ for some φ in $L^q(-\pi, \pi)$ [2].

REFERENCES

1. W. G. Bade and P. C. Curtis, Jr., *Embedding theorems for commutative Banach algebras*, Pacific J. Math. **18** (1966), 391—409. MR **34** #1878.

2. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954. MR **16**, 914.

3. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341—366. MR **13**, 839.

4. T. Kawata, *A relation between the theories of Fourier series and Fourier transforms*, Proc. Imp. Acad. Tokyo **16** (1940), 255—261. MR **2**, 94.

5. R. M. Young, *Interpolation in a classical Hilbert space of entire functions*, Trans. Amer. Math. Soc. **192** (1974), 97—114.

6. ———, *An extension of the Hausdorff-Young theorem*, Proc. Amer. Math. Soc. **45** (1974), 235—236.

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