RESIDUAL EQUISINGULARITY

JOSEPH BECKER AND JOHN STUTZ

ABSTRACT. Let $V$ be a complex analytic set and $S_g V$ the singular set of $V$ be in codimension one; then the set of points of $S_g V$ for which $V$ is not residually equisingular along $S_g V$ is a proper analytic subset of $S_g V$. $V$ is said to be residually equisingular along $S_g V$ if all one dimensional slices of $V$ transverse to $S_g V$ have isomorphic resolutions.

Let $V$ be an analytic subset of a domain $U \subseteq \mathbb{C}^n$ of pure dimension $r$ and let $S_g V$, the singular locus of $V$, be of dimension $r - 1$. For $p \in S_g V$ such that near $p$, $S_g V$ is an $(r - 1)$-dimensional manifold, the second author introduced [3] three different notions of equisingularity of $V$ along $S_g V$ at $p$: weak, strong, and residual. These generalize various portions of the theory of equisingularity for hypersurfaces developed by Zariski [5]. The results pertaining to residual equisingularity were published jointly in [1]. The purpose of this note is to improve one of the results in [1]. There we showed that if $V$ is weakly equisingular along $S_g V$ at $p$, then near $p \{ q \in S_g V : V$ is not residually equisingular along $S_g V$ at $q \}$ is contained in an analytic subset of $S_g V$ of dimension $< r - 1$. We improve this showing that $\{ q \in S_g V : V$ is not residually equisingular along $S_g V$ at $q \}$ is an analytic subset of $S_g V$ of dimension $< r - 1$. In the course of our proof we will clarify our original definition of residual equisingularity and develop an alternative formulation which does not depend on the imbedding of $V$ in $U$.

The authors would like to thank Professor Joseph Lipman for suggesting the technique employed here.

Let $p$, $S_g V$ and $V$ be as above. Replacing $U$ by a suitable neighborhood of $p$ we may select coordinates $(x_i)$ such that $p$ becomes the origin, $S_g V = \{ q \in U : q_i = 0$ for $i \geq r \}$, and for $q \in S_g V$, $V_q = \{ x \in V : x_i = q_i$ for $i < r \}$ is one dimensional. Such coordinates will be said to be adapted to $V$ at $p$. Each $V_q$ can be resolved by a sequence of local quadratic transformations, however the series may vary greatly with $q$; $V$ is said to be residually equisingular (see Definitions 1 and 2) if the series does not vary. For such coordinates, and for $W$, $W'$ appropriate open subsets of $V$, we define 3 types of maps from $W$ to $W'$:

(i) $T(x) = (x_1, \ldots, x_r, x_{r+1} - t_1(x), \ldots, x_n - t_{n-r}(x))$ where the $t_i$ are holomorphic functions of $x_1, \ldots, x_r$ alone.

(ii) $S(x) = (x_1, \ldots, x_{r-1}, s_0(x), \ldots, s_{n-r}(x))$ where $s_i(x) = \sum_{j=r}^n a_{ij} x_j, a_{i,j} \in \mathbb{C}$, and $rk(a_{i,j}) = n - r$. 

Received by the editors March 12, 1975.

AMS (MOS) subject classifications (1970). Primary 32C45; Secondary 14B05.

Key words and phrases. Residual equisingularity, local quadratic transformation, analytic variety, resolution.
(iii) \( Q(x) = (x_1, \ldots, x_r, x_rx_{r+1}, \ldots, x_rx_n). \)

**Definition 1.** A connected sequence of local quadratic transformations is a sequence of maps \( f_i: W_i \to W_{i-1}, i = 1, \ldots, l \), such that:

(a) \( f_l = f_1 \circ f_2 \circ \cdots \circ f_l \) is a proper modification of \( W_0 \), that is \( f_l^{-1}(\text{compact set}) \) is compact and \( f_l \) is biholomorphic on the inverse image of \( \text{Reg } W_0 \), the regular points of \( W_0 \), and \( f_l^{-1}(\text{Reg } W_0) \) is dense in \( W_l \).

(b) \( f_i \) is the restriction of a map of type (i), (ii) or (iii) on each component \( W_{j,k} \) of \( W_j \).

(c) If \( f_j(W_{j,k_1}) \cap f_j(W_{j,k_2}) \) is nonnull then \( f_j|W_{j,k_1} \) and \( f_j|W_{j,k_2} \) are of the same type.

Suppose that a connected sequence of quadratic transformations is given and the image of \( f_j \) is a neighborhood of \( p \). Set \( V_j = f_j^{-1}(V) \) (of course if a type (iii) map occurs, we omit those components which are contained in \( \{ x \in U: x_r = 0 \} \)).

**Definition 2.** \( V \) is residually equisingular along \( \text{Sg } V \) at \( p \) if there is a connected sequence of quadratic transformations as above such that:

(a) \( f_l: V_l \to V \) is a resolution;

(b) \( f_l^{-1}(\text{Sg } V) \) is an \( r-1 \) manifold and \( f_l \) restricted to each component is biholomorphic.

The above definition appears weaker than that given in [1]. There we required that given any \( q \in f_0^{-1}(p) \) and \( V_{0,k} \) a branch of \( V_0 \) at \( q \) that either \( V_{0,k} \) is nonsingular at \( q \) or \( \text{Sg } V_{0,k} = f_0^{-1}(\text{Sg } V) \) near \( q \). This condition is actually a consequence of our other conditions. To see this assume there is a \( q \) at which \( \text{Sg } V_{0,k} \) is a proper subset of \( f_0^{-1}(\text{Sg } V) \); then \( \dim \text{Sg } V_{0,k} < r-1 \). Since the maps \( (f_j)^{-1} \) are weakly holomorphic the singular loci of the portions of the \( V_j, j > j_0 \), lying over \( V_{0+1,k} \) are also of dimension \( < r-1 \). Thus it suffices to consider the case where \( V_{0+1,k} \) is nonsingular and \( f_{j_0+1}|V_{0+1,k} \) is of type (iii). Now as in the first portion of the proof of Proposition 1 in [1], one sees that \( \dim C_4(V_{0+1,k}, q) = r \); and [2, Proposition 1.8] \( \dim C_4 V = \dim V, \) \( \text{codim } \text{Sg } V \geq 2 \) implies \( V \) nonsingular.

**Definition 3.** Let \( X \) be an analytic variety and \( \mathcal{I} \) a coherent sheaf of ideals in \( X_0 \), the reduced sheaf of holomorphic functions on \( X, \mathcal{O}/I(V) \) where \( I \) is self-radical. Then the blow-up of \( X \) along \( \mathcal{I}, B_j(X) \), is defined as follows: let \( g_1, \ldots, g_m \) generate \( \mathcal{I} \) over \( X_0, Y = \text{support } \mathcal{I}, \) and \( B_j(X) \) the closure in \( X \times \mathbb{CP}^{m-1} \) of \( \{(x, z) \in (X - Y) \times \mathbb{CP}^{m-1}: z_ig_j(x) = z_jg_i(x) \text{ all } i,j \} \). Then the natural projection \( \pi: B_j(X) \to X \) is a proper modification of \( X \). This construction is independent (up to analytic isomorphism) of the choice of generators of \( \mathcal{I} \) and of the coordinates on \( \mathbb{C}^n \).

Now let \( \mathcal{I} \) be the ideal sheaf of \( \text{Sg } X \) and \( B_s(X) \) denote the blow-up of \( X \) along \( \mathcal{I} \). We will see there is a one-to-one correspondence between local quadratic transforms of \( X \) and blow-ups such that \( \pi^{-1}(\text{Sg } X) \to \text{Sg } X \) is a biholomorphism on each component.

Let \( p \in \text{Reg } (\text{Sg } (X)), (x_i) \) coordinates adapted to \( X \) at \( p \), and \( Q: X' \to X \) a local quadratic transform of the third type \( Q(x_1, \ldots, x_n) = (x_1, \ldots, x_r, x_rx_{r+1}, \ldots, x_rx_n) = (y_1, \ldots, y_n). \) Now \( I(\text{Sg } V) \) is generated by \( x_r, x_{r+1}, \ldots, x_n \), so form the blow-up \( B_s(X) \subset \{(y, z) \in X \times \mathbb{CP}^{n-1}: y_iz_j = y_jz_i \text{ for } i, j \geq r \} \). There is an injection \( i: X \to B_s(X), \)

\[
i(x_1, \ldots, x_n) = (x_1, \ldots, x_r, x_rx_{r+1}, \ldots, x_rx_n; 1, x_{r+1}, \ldots, x_n) 
\]
such that $Q = \pi i$. Since $B_3(X) \subset \{(y,z): z_r \neq 0\}$, $\beta: B_3(X) \to X'$, $\beta(x,y) = x$ is the inverse to $i$. Since $Q$ is unramified over $SgX$, $\pi$ is also.

Conversely if $\pi: B_3(X) \to X$ is unramified over $SgX$, one can define an inverse $\varphi: SgX \to B_3(X)$ of $\pi$ and by shrinking the neighborhood of $p$, assume that some $Z_\varepsilon$ (say $Z_\varepsilon$) is nonzero on the image of $\varphi$. Let

$$Q(y_1, \ldots , y_n) = (y_1, \ldots , y_r, y_{r+1}, \ldots , y_r y_{r+1})$$

and $X' = Q^{-1}(X)$; there is an analytic equivalence $\rho: B_3(X) \to X'$ defined by

$$\rho(x_1, \ldots , x_n, z_r, \ldots , z_n) = (x_1, \ldots , x_r, z_{r+1}/z_r, \ldots , z_n/z_r)$$

such that $\pi = \varphi \rho$. $Q$ is unramified over $Sg V$, since $\pi$ is too.

For each integer $j > 0$ we can define a space $B_j$ and a map $g_j: B_j \to V$ as follows: $\pi_j: B_j \to V_j$ is the blow-up of $V$ along $Sg V_j$. For $j > 1$, set $S_j = g_j^{-1}(Sg V)$, let $\pi_j: B_{j+1} \to B_j$ be the blow-up of $B_j$ along $S_j$ and set $g_{j+1} = \pi_j \circ g_j$.

**Lemma.** The following are equivalent:

(a) $V$ is residually equisingular along $Sg V$ at $p \in Sg V$.

(b) If one replaces $U$ by a suitable neighborhood of $p$, then there is an integer $m > 0$ such that $B_m$ is nonsingular, and for $j < m$, $S_j$ is an $(r - 1)$-dimensional manifold, and $\pi_j$ restricted to each component of $S_j$ is biholomorphic.

**Proof.** Follows immediately from the above and the fact that types (i) and (ii) maps are biholomorphic.

**Proposition.** The set $A$ of points $p$ such that $V$ is not residually equisingular along $Sg V$ at $p$ is an analytic subset of $V$.

**Proof.** Take a succession of blow-ups of $V$ along the singular locus; define inductively $V_0 = V$, $V_{i+1} = B_{Sg V_i}(V_i), \pi_i: V_i \to V_{i-1}, g_j = \pi_1 \circ \cdots \pi_j$ and $S_i = g_r^{-1}(Sg V)$. Since $A$ is a nowhere dense set in $Sg V$ [1, Proposition 2], there exists some $l > 0$ such that codim$_{S_j} Sg V_l \geq 2$. Then, we have that

$$A = g_l(Sg B_l) \cup \left( \bigcup_j g_j(Sg S_j \cup \{ q \in S_j - Sg S_j: rk_q g_j|S_j < r - 1 \}) \right)$$

is the finite union of images of analytic sets under proper maps.

In addition to providing a characterization of residual equisingularity, the Lemma also provides the justification for a simple method of testing for residual equisingularity in particular examples. As observed in [3], since residual equisingularity implies weak equisingularity, all potential equisingular points have Puiseux series normalizations, so one can attempt to check for residual equisingularity by direct calculation. For example let $V$ be the image of the map $f: \mathbb{C}^2 \to \mathbb{C}^4$ given by $f(x,y) = (x, y^9, y^7, xy^8)$. (See [1, Example 2] for more details on this example.) The obvious attempt to construct the required connected sequence of local quadratic transformations leads to a dead end. However, initially one does not know that some other sequence will not work. By constructing from the initial sequence the blow-ups $B_1, B_2$, one sees that $\dim \pi^{-1}_2(0) = 1$, so $V$ is not residually equisingular at 0.
In the classical theory of plane curves, one has the equivalence of characteristic pairs and multiplicities $e_i$ of the canonical sequence of blow-ups which resolve the singularity of an irreducible plane curve. These determine the length of the conductor $\dim \mathcal{O}/\mathcal{O}$, and the, exponent of the conductor on the normalization $\dim \mathcal{O}/\mathcal{O}$, via the Italian Geometers formula:

$$\exp = 2 \cdot \text{length} = \sum e_i(e_i - 1).$$

For equisingularity in codim one [5], these results extend as follows: $V$ is equisingular along $\text{Sg } V$ if and only if the curves have the same multiplicity sequence. Furthermore, if the length of the conductor is the same for each curve, then $V$ is equisingular along $\text{Sg } V$.

For equisingularity in higher codim, this all breaks down. Neither the exponent or length is constant, and it is not true that one thing is twice the other.

**Example.** Let $V$ be the image in $\mathbb{C}^4$ of $(s, t) \rightarrow (s, t^3, t^4, st^5)$. Then $V$ is residually and strongly equisingular. For $s \neq 0$, exponent = 3, length = 2, and $\sum e_i(e_i - 1) = 6$. For $s = 0$, exponent = 6, length = 3, and $\sum e_i(e_i - 1) = 6$.

However, one can easily see from looking at the Puiseux expansions the multiplicity sequence is the same for each curve in a residually equisingular variety and so $\sum e_i(e_i - 1)$ is constant. Recently Fischer [4] has found a possible interpretation for this number: let $C$ be the complete local ring of a irreducible curve, $\overline{C}$ its integral closure in its field of quotients, and $I_C$ the high order differentials of $\overline{C}$ over $C$. Then length $I_C = \sum e_i(e_i - 1)$ and $I_{C_1} \simeq I_{C_2}$ as $\overline{C}$ modules if and only if $C_1$ and $C_2$ have the same multiplicity sequence.

**Remark.** Strong equisingularity does not even imply that the multiplicity sequence is constant. This can be seen from the example $(s, t) \rightarrow (s, t^5, t^7, st^8)$.

**References**


Division of Mathematical Sciences, Purdue University, West Lafayette, Indiana 47907

Allen Center, State University of New York at Albany, Albany, New York 12222

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use