LIE GROUPS ISOMORPHIC TO DIRECT PRODUCTS OF UNITARY GROUPS

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Abstract. A criterion is given for a compact connected subgroup of \( \text{Gl}(n, \mathbb{C}) \) to be isomorphic to a direct product of unitary groups. It implies that a compact connected subgroup of rank \( n \) in \( \text{Gl}(n, \mathbb{C}) \) is isomorphic to a direct product of unitary groups.

The paper gives a generalization of some of the results in [3]. Let \( G \) be a compact connected subgroup of \( \text{Gl}(n, \mathbb{C}) \). We denote by \( L(G) \) the Lie algebra of \( G \) and set \( H(G) = \text{i}L(G) \). The rank of \( G \) is the dimension of a maximal torus in \( G \) (see [1, p. 93]).

Theorem. Let \( G \) be a compact connected subgroup of rank \( k \) in \( \text{Gl}(n, \mathbb{C}) \). Suppose there exist \( r > k \) orthogonal idempotents \( a_1, \ldots, a_r \) in \( H(G) \). Then \( r = k \) and \( G \) is isomorphic (as a Lie group) to a direct product of unitary groups:

\[ G \cong U(n_1) \times \cdots \times U(n_m) \text{ with } n_1 + \cdots + n_m = k. \]

Proof. By [2, p. 176, Theorem 1] \( G \) is similar to a subgroup of \( U(n) \). Hence we may assume that \( G \) is a subgroup of \( U(n) \). Thus the operators in \( H(G) \subset \text{End}(\mathbb{C}^n) \) are hermitian. Since \( a_1, \ldots, a_r \) commute we see that \( T = \{ \exp(it_1a_1 + \cdots + it_ra_r) | t_1, \ldots, t_r \in \mathbb{R} \} \) is a torus in \( G \) of dimension \( r \). Clearly \( r = k \) and \( T \) is a maximal torus. If \( a \in H(G) \) then \( \exp(ita) \in G \) and is contained in some conjugate of \( T \) (see [1, p. 89]), i.e. \( \exp(ita) \in u^{-1}Tu = u^*Tu \) for some \( u \in G \). It follows that \( a = t_1u^*a_1u + \cdots + t_ru^*a_ru \). Since \( a^2 = t_1^2u^*a_1^2u + \cdots + t_r^2u^*a_r^2u \) and \( u^*a_su \in H(G) \) for \( s = 1, \ldots, r \) we see that \( a^2 \in H(G) \). Let \( b \in H(G) \), too. Since \( ab + ba = (a + b)^2 - a^2 - b^2 \) we see that \( ab + ba \in H(G) \). Also, \( ab - ba \in iH(G) \) since \( ia, ib \in L(G) \). Thus \( ab \in H(G) + iH(G) \). Let \( A(G) = H(G) + iH(G) \). It follows that \( A(G) \) is an algebra. Clearly, it is a finite dimensional C*-algebra. By the Wedderburn decomposition there exist central idempotents \( e_1, \ldots, e_m \in A(G) = A \) such that \( A = Ae_1 \oplus \cdots \oplus Ae_m \) and \( Ae_s \) is isomorphic to \( \text{End}(X_s) \) for some finite dimensional vector space \( X_s \) over \( \mathbb{C} \) (\( s = 1, \ldots, m \)).

The ideal \( Ae_1 \) is closed, hence selfadjoint and a C*-subalgebra of \( A \). Clearly, \( e_1 \) is the identity on \( Ae_1 \) and hence \( e_1^* = e_1 \). Consider the group \( V \) of unitary elements in \( Ae_1 \). The isomorphism \( Ae_s \cong \text{End}(X_s) \) defines a (continuous) representation of \( V \) on \( X_s \). Using once more [2, p. 176, Theorem 1] we equip \( X_s \) with an inner product such that the isomorphism maps \( V \) into the unitary...
group of \( \mathcal{L}(X_s) \), the C*-algebra of all linear operators on the Hilbert space \( X_s \). Consequently, hermitian elements in \( A_{es} \) are mapped into hermitian operators and our isomorphism in an isometric*-isomorphism. We identify the algebras \( A_{es} \) and \( \mathcal{L}(X_s) \) in this sense.

Since \( \exp: L(G) \rightarrow G \) is surjective, \( G \subseteq A \). If \( u \in G \) then \( (ue_s)^* ue_s = e_s u^* u e_s = e_s \). Thus \( ue_s \) is a unitary operator on \( X_s \). Consider the smooth homomorphism \( G \rightarrow U(X_1) \times \cdots \times U(X_m) \) given by \( u \mapsto (ue_1, \ldots, ue_m) \) (\( U(X_j) \) denotes the unitary group on \( X_j \)). We claim this homomorphism is onto. Let \( u_l \in U(X_1) \). There exists a hermitian element \( h_l \in Ae_l \) such that \( \exp(ih_l) = u_l \). Consider \( h_l \) as an element in \( A \). Then \( \exp(ih_l) = (u_l, 1, \ldots, 1) \). Observe that the inverse \( (ue_1, \ldots, ue_m) \mapsto ue_1 + \cdots + ue_m \) is also smooth and that \( \text{rank}(U(n_1) \times \cdots \times U(n_m)) = n_1 + \cdots + n_m \).

**Corollary.** Let \( G \) be a compact connected subgroup of rank \( n \) in \( \text{Gl}(n, \mathbb{C}) \). Then \( G \) is isomorphic (as a Lie group) to a direct product of unitary groups.

**Proof.** As before, we may assume that \( G \leq U(n) \). Let \( T \) be a maximal torus in \( G \). Then \( iL(T) \) contains \( n \) commuting linearly independent hermitian operators, say \( h_1, \ldots, h_n \). It is well known that these operators have a common orthogonal eigenbasis. Thus there exist \( s \leq n \) orthogonal projections \( p_1, \ldots, p_s \) such that every \( h_i \) is a linear combination of \( p_1, \ldots, p_s \). Since \( h_1, \ldots, h_n \) are linearly independent, \( s = n \). Thus \( iL(G) \) contains \( n \) orthogonal idempotents and we may use the Theorem.

**References**


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