CONFIGURATION-LIKE SPACES AND THE BORSUK-ULAM THEOREM

FRED COHEN AND EWING L. LUSK

Abstract. Some extensions of the classical Borsuk-Ulam Theorem are proved by computing a bound on the homology of certain spaces similar to configuration spaces. The Bourgin-Yang Theorem and a generalization due to Munkholm are special cases of these results.

1. Introduction. The purpose of this paper is to extend and unify several generalizations of the Borsuk-Ulam Theorem. Let $\pi_p$ denote the cyclic group of prime order $p$ and let $X$ be a pathwise connected Hausdorff space on which $\pi_p$ acts freely. Suppose that $M$ is some fixed manifold and that $f: X \to M$ is any map. We are interested in conditions on $X$, depending on $M$ but not on $f$, which are sufficient to insure that a certain number of points in some orbit are sent to the same point in $M$ by $f$. Specifically, let $\sigma$ denote the generator of $\pi_p$ and define

$$A(f, q) = \{x \in X | \text{there exist } i_1, i_2, \ldots, i_q \text{ with } 0 \leq i_1 < i_2 < \cdots < i_q < p \text{ and } f(\sigma^{i_1}x) = f(\sigma^{i_2}x) = \cdots = f(\sigma^{i_q}x)\}.$$ 

In the case $M = \mathbb{R}^n$, we prove the following, in which $\dim A$ denotes the covering dimension of $A$, and all cohomology is taken with $\mathbb{Z}_p$ coefficients unless otherwise stated.

**Theorem 1.** If $H^i(X) = 0$ for $0 < i < (n - 1)(p - 1) + q - 1$ and $q \geq \frac{1}{2}(p + 1)$ or $q = 2$, then $A(f, q) \neq \emptyset$.

**Theorem 2.** If $X$ is a $\mathbb{Z}_p$-orientable $m$-manifold and $H^i(X) = 0$ for $0 < i < (n - 1)(p - 1) + q - 1$ and $q \geq \frac{1}{2}(p + 1)$ or $q = 2$, then $\dim A \geq m - (n - 1)(p - 1) - q + 1$.

Special cases of these theorems are known:

1. The classical Borsuk-Ulam Theorem is Theorem 1 with $X = S^n$ and $q = p = 2$ [1].
2. The “mod $p$ Bourgin-Yang Theorem” of Munkholm is Theorem 2 with $q = p$ and $X$ a mod $p$ homology $m$-sphere [6]. For this special case the proof given below is much simpler than Munkholm’s.
3. The case $q = 2$ of Theorem 1 appears in [3].

Theorems 1 and 2 are actually special cases of a more general theorem, in which $\mathbb{R}^n$ is replaced by an arbitrary manifold $M$. That is, for each $M$ and

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There is a number $N(M, p, q)$ (defined below) such that for $f: X \to M$ we have:

**Theorem 3.** If $H^i(X) = 0$ for $0 < i \leq N(M, p, q)$, then $A(f, q) \neq \emptyset$. If in addition we assume that $X$ is a $\mathbb{Z}_p$-orientable $m$-manifold, then $\dim A(f, q) \geq m - N(M, p, q) - 1$.

To define the numbers $N(M, p, q)$, consider the subspace $G(M, p, q)$ of $(M)^p$ consisting of the $p$-tuples in which no $q$ coordinates coincide. More precisely,

$$G(M, p, q) = \{ (x_1, \ldots, x_p) \mid \text{for any } (x_{i_1}, \ldots, x_{i_q}) \text{ with } 0 < i_1 < \cdots < i_q \leq p, \text{ at least 2 of the } x_{i_j} \text{'s are different} \}. $$

Note that $G(M, p, q) \subset G(M, p, q + 1)$, $G(M, p, p) = (M)^p - \Delta_M$, and $G(M, p, 2)$ is the Fadell-Neuwirth configuration space [5]. The group $\pi_p$ acts freely on $G(M, p, q)$ by cyclic permutation of coordinates, and the inclusions $G(M, p, q) \subset G(M, p, p)$ are equivariant. For some large $n$, $G(M, p, q)$ embeds in $G(R^n, p, q)$ via the embedding of $M$ in $R^n$. Define

$$G(R^\infty, p, q) = \lim\limits_n G(R^n, p, q).$$

**Proposition.** $G(R^\infty, p, p)$ is a free $\pi_p$-space with trivial homotopy groups and hence $G(R^\infty, p, p)/\pi_p$ is a $K(\pi_p, 1)$.

**Proof.** Since

$$H_*(G(R^\infty, p, p); \mathbb{Z}) = H_*(\lim\limits_n G(R^n, p, p); \mathbb{Z}) = \lim\limits_n H_*(G(R^n, p, p); \mathbb{Z})$$

and $G(R^n, p, p) \approx S^{n(p-1)-1}$, $G(R^\infty, p, p)$ has trivial homology groups. Since $G(R^\infty, p, p)$ is simply connected, the result follows from the Hurewicz Theorem.

**Definition of $N(M, p, q)$**. Let $\phi$ be an equivariant embedding of $G(M, p, q)$ in $G(R^\infty, p, p)$. Recall that $H^iK(\pi_p, 1) = \mathbb{Z}_p$ for all $i$ and define $N(M, p, q)$ to be the largest $N$ such that $\phi^*$ is not the zero homomorphism. We have not calculated $N(M, p, q)$ for $M \neq R^n$ except for the case $q = 2$ (see [4]). When $M = R^n$, it is sufficient to calculate the first nonvanishing homology class in a certain union of spheres. This we do in §3. The result is:

**Theorem 4.** $N(R^n, p, q) \leq (n-1)(p-1) + q - 2$ if $q \geq \frac{1}{2}(p+1)$ or $q = 2$.

2. **Proofs of Theorems 1, 2, and 3**. We prove Theorem 3. Theorems 1 and 2 follow immediately from Theorems 3 and 4. Let $\sigma$ be the generator of $\pi_p$ and define $\psi: X \to (M)^p$ by $\psi(x) = (f(x), f(\sigma x), \ldots, f(\sigma^{p-1} x))$. If $A(f, q) = \emptyset$ then $\psi$ is an equivariant map of $X$ into $G(M, p, q)$. Consider the following diagram, in which the vertical arrows represent projections.

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & G(M, p, q) \\
\downarrow & & \downarrow \\
X/\pi_p & \xrightarrow{\phi} & G(R^\infty, p, p)/\pi_p \\
\end{array}
$$

If $H^i(X) = 0$ for $0 < i \leq N(M, p, q)$, then it follows from the naturality of
the spectral sequence for a covering that $\mathbin{\hat{\phi} \hat{\psi}}^*$ is a monomorphism in degrees less than or equal to $N(M, p, q) + 1$, contradicting the fact that $\mathbin{\hat{\phi}^*} = 0$ in degrees greater than $N(M, p, q)$. This proves the first part of the theorem.

Now suppose that $X$ is a $\mathbb{Z}_p$-orientable $m$-manifold. Observe that $\psi$ restricts to an equivariant map of $X - A(f, q)$ into $G(M, p, q)$ and that we may assume $X - A(f, q)$ is path connected. By the above argument there must be some $j$, $0 < j \leq N(M, p, q)$, such that $H^j(X - A(f, q)) \neq 0$, and hence

$$H_j(X - A(f, q)) \neq 0.$$  

By Alexander Duality, $H^{m-j}(X, A(f, q)) \neq 0$. Similarly $H_j(X) = 0$ implies $H^{m-j}(X) = 0$, so by the exact cohomology sequence

$$\mathbin{\check{H}^{m-j-1}(A(f, q))} \neq 0.$$  

By the argument which appears in [6], this is enough to prove that the covering dimension of $A(f, q)$ is greater than or equal to $m - N(M, p, q) - 1$.

3. **Proof of Theorem 4.** First we remark that the case $p = q$ is particularly simple since $G(R^n, p, p) = (R^n)^p - \Delta \simeq S^{n(p-1)-1}$, and so $N(R^n, p, p) \leq n(p - 1) - 1$. The case $q = 2$ appears in [4]. In general, we proceed as follows. The standard strong deformation retraction of $R^{np} - \{0\}$ onto $S^{np-1}$ restricts to a strong deformation retraction of $G(R^n, p, q)$ onto its intersection with $S^{np-1}$. Let $K(n, p, p - q)$ denote the complement of the image of $G(R^n, p, q)$ under this deformation. We let $k = p - q$ and note that $K(n, p, k)$ is the union of spheres of dimension $n(k + 1) - 1$. Our method of bounding $N(R^n, p, q)$ will be the rather crude one of bounding $H^*(G(R^n, p, q))/\mathbb{Z}_p$. In general we will do this by finding a lower bound for $H_\ast K(n, p, k)$ using the Mayer-Vietoris sequence and then applying Alexander Duality in the $(np - 1)$-sphere.

First we need some notation for the pieces of $K(n, p, k)$ to which we will apply Mayer-Vietoris. Let $I = (i_1, \ldots, i_j)$, $j \leq k$, denote any $j$-tuple of integers with $0 < i_1 < i_2 < \cdots < i_j \leq p$. We define the length of $I$ to be $j$ and denote it by $l(I)$. We also permit $I$ to be empty and in this case define $l(I) = 0$. Now let $m$ be any positive integer less than or equal to $p - k$ and define

$$W(I, k, m) = \{(x, x, \ldots, x, y_1, x, x, \ldots, x, y_2, x, x, \ldots, x, y_k, x, x, \ldots, x)\mid$$

$$x \in R^n, y_s \text{ occurs in the } i_s \text{ th place for } s = 1, 2, \ldots, j,$$

and there are $mx$ 's between $y_j$ and $y_{j+1}\}.$

That is, the coordinates which are not specified to be equal to other coordinates occur in places $i_1, i_2, \ldots, i_j, i_j + m + 1, \text{ and beyond. By abuse of notation we write the sequence } x, x, \ldots, x (a_1 \text{ terms}) \text{ as } x^{a_1}. \text{ A typical point in } W(I, k, m) \text{ looks like}$

$$(x^{a_1}y_1x^{a_2}y_2 \cdots x^{a_j}y_jx^{m}y_{j+1}x^{l_j}y_{j+2}x^{l_j+2} \cdots y_kx^{l_{k-j}}}),(\text{ where } a_1 + \cdots + a_j + m + l_1 + \cdots + l_{k-j} = p - k = q). \text{ We note that the}$
\( \alpha_i \)'s are determined by \( I \) and ignore them. Observe that \( W(I, k, m) \) is a union of equatorial \((n(k + 1) - 1)\)-spheres in \( S^{np-1} \). We assume that \( q \geq \frac{1}{2}(p + 1) \).

**Lemma 1.** \( \bigcup_{i=0}^{r} W(I, k, i) \cap W(I, k, m + 1) = W(I, k - 1, m + 1) \).

**Proof.** Observe that a point is in the left-hand side if and only if \( y_{j+1} = x \). Therefore \( y_{j+2}, \ldots, y_k \) can be relabeled \( y_{j+1}, \ldots, y_{k-1} \).

**Lemma 2.** \( H_i W(I, k, m) = 0 \) if \( 0 < i < n + k - 1 \).

**Proof.** The proof is by induction on \( k \) and for fixed \( k \) by downward induction on \( l(I) \). The lemma is true for \( k = 0 \) since \( W(I, 0, m) \) is an \((n - 1)\)-sphere. Fix \( k \) and assume that the lemma is true with \( k - 1 \) replacing \( k \). The induction on \( l(I) \) starts with \( l(I) = k \). In this case \( W(I, k, m) \) is an \((n(k + 1) - 1)\)-sphere, so the lemma is true. Now suppose that \( l(I) = R - 1 \) and that the lemma is true for all \( I \) with \( k \geq l(I) \geq R \). A point

\[
x^{a_1} y_1 \cdots y_{R-1} x^m (y_{R+1} x^i \cdots x^{R-1-k})
\]

can be rewritten as

\[
x^{a_1} y_1 \cdots y_{R-1} x^i (y_{R+1} x^{i+1} \cdots x^{R-1-k})
\]

so we have \( W(I, k, m) = \bigcup_t W(J, k, t) \), where \( t \) varies from 0 to some number \( s \) determined by \( I \), \( k \), and \( m \). Since \( l(J) > l(I) \), \( H_i W(J, k, r) = 0 \) for \( 0 < i < n + k - 1 \) and all \( r \) by induction. Now we assume that \( H_i (\bigcup_{t=0}^{r} W(J, k, t)) = 0 \) for \( 0 < i < n + k - 1 \) and show that \( H_i (\bigcup_{t=0}^{r+1} W(J, k, t)) = 0 \) for \( 0 < i < n + k - 1 \). By Lemma 1 the Mayer-Vietoris sequence is

\[
\cdots \to H_i \left( \bigcup_{t=0}^{r} W(J, k, t) \right) \oplus H_i W(J, k, r + 1) \to H_i \left( \bigcup_{t=0}^{r+1} W(J, k, t) \right) \to H_{i-1} W(J, k - 1, r + 1) \to \cdots
\]

in which the left side is 0 for \( 0 < i < n + k - 1 \) by the inductions on \( r \) and on \( l(I) \) and the right side is 0 for \( 0 < i < n + k - 1 \) by the induction on \( k \). Therefore \( H_i W(I, k, m) = H_i W(J, k, t) = 0 \) for \( 0 < i < n + k - 1 \).

**Remark.** Note that the second half of this proof shows that for any sequence \( J \), if \( H_i W(J, k, t) = 0 \) for \( 0 < i < n + k - 1 \) and all \( t \), then \( H_i \bigcup_{t=0}^{s} W(J, k, t) = 0 \) for \( 0 < i < n + k - 1 \), for any \( s \).

**Proof of Theorem 4.** First we note that \( K(n, p, p - q) = \bigcup_{t=0}^{p} W(\varnothing, k, t) \), so by the above remark \( H_i K(n, p, p - q) = 0 \) for \( 0 < i < n + p - q - 1 \). By the remarks at the beginning of this section

\[
H^i G(R^n, p, q) \cong H^i (S^{np-1} - K(n, p, p - q)),
\]

which is in turn isomorphic to \( H^i_{np-1-i} (S^{np-1}, K(n, p, p - q)) \) by Alexander Duality. Then by the exact sequence

\[
\cdots \to H^i_{np-1-i} (S^{np-1}) \to H^i_{np-1-i} (S^{np-1}, K(n, p, p - q)) \to H^i_{np-1-i} (K(n, p, p - q)) \to \cdots
\]

we have that \( H^i G(R^n, p, q) = 0 \) for \( i > (n - 1)(p - 1) + q - 2 \). This is suf-
ficient, by the argument in [4], for example, to conclude that

\[ H^i(G(R^n, p, q)/\mathbb{Z}_p) = 0 \quad \text{for } i > (n - 1)(p - 1) + q - 2. \]

4. An example. In some situations these results are best possible ones. For example, in [6] Munkholm gives an example for each odd \( p \) and each \( m \) of a \( \pi_p \)-action on \( S^{m(p-1)-1} \) and a map from \( S^{m(p-1)-1} \) to \( R^m \) such that no entire orbit is sent to the same point in \( R^m \). Our Theorem 1 states that there is an orbit in which \( p - 1 \) points are sent to the same point. This example shows that in the case \( q = p - 1 \) one can have \( H^i(X) = 0 \) for \( 0 < i < (n - 1) \cdot (p - 1) + q - 1 \) with \( A(f, q + 1) = \emptyset \).

Remark. We conjecture that Theorem 4 and hence Theorems 1 and 2 are true without the restriction \( q \geq \frac{1}{2}(p + 1) \), although it is not hard to see that Lemma 1 and hence our method of proof break down if \( q < \frac{1}{2}(p + 1) \). The difficulty can be seen in the case \( l = \emptyset, p = 5, q = 2, m = 0 \). Lemma 1 then says \( W(\emptyset, 3, 0) \cap W(\emptyset, 3, 1) = W(\emptyset, 2, 1) \). However a point of the form \((x, z, z, y, x)\) is in the left-hand side but not the right.

REFERENCES


DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DEKalb, ILLINOIS 60115