INTEGRAL CLOSURES OF UNCOUNTABLE
COMMUTATIVE REGULAR RINGS

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Abstract. Necessary and sufficient conditions are given for a commutative
regular ring to have a prime integrally closed extension.

In this paper we give necessary and sufficient conditions for a commutative
regular ring $R$ to have a prime integral closure. In [1] it was shown that for a
commutative regular ring $R$ to have a prime integral closure, it is necessary
that every polynomial $p(x)$ in $R[x]$ have an unambiguous factor (see definitions
below), and that in the case that $R$ is countable this condition is also sufficient.
An example was given to show that this condition is not sufficient if $R$ is
uncountable. It was also seen in [1] that if $R$ has a prime integral closure, then
this closure is unique. I would like to thank Bonnie Gold and Gadi Moran for
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Definitions. (1) $K_{CR}$ is the theory of commutative regular rings;

$$K_{CR} = K_{CR} \cup \{\text{every monic polynomial has a root}\}$$

is the theory of integrally closed commutative regular rings.

(2) If $R \models K_{CR}$ and $p(x) \in R[x]$, we call $p(x)$ unambiguous if on no nonzero
idempotent $e$ is it the case that $p(x) = u(x)v(x)$ with $(u(x), v(x)) = 1$ on $e$. (An
identity holds on $e$ if it holds in $Re$.) This condition is equivalent to $p(x)$ being
a power of an irreducible polynomial at every point of $S_R$, the Stone space of
$R$ (= Spec $(R)$).

$$T = K_{CR} \cup \{\text{every polynomial has an unambiguous factor}\}.$$ 

(3) If $R \models K_{CR}$ and $R \subset \bar{R} \models K_{CR}$, we call $\bar{R}$ a prime extension of $R$ to a
model of $K_{CR}$, or an (in fact the) integral closure of $R$ if whenever
$f: R \to R_1 \models K_{CR}$ is an embedding, $f$ extends to an embedding of $\bar{R}$ into $R_1$. If
we drop the condition that $\bar{R} \models K_{CR}$ we call $\bar{R}$ a prime extension of $R$.

(4) If $R \models K_{CR}$ and $R \subset \bar{R} \models K_{CR}$, we call $\bar{R}$ sequentially prime over $R$ if
$\bar{R} = \bigcup_{a<\lambda} \bar{R}_a$ with $R_0 = R$, $R_\delta = \bigcup_{a<\delta} R_a$ for limit ordinals $\delta < \lambda$ and
$R_{a+1} = R_a[a_a]$, with $a_a$ a root of an unambiguous polynomial $p_a(x) \in R_a[x]$.
(In other words, $\bar{R}$ can be realized as a sequence of one element extensions,
each prime over the previous ones—see [1].)

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Let $R \subseteq K_{CR}$. We call $R$ thin if there is a set $\mathcal{P} \subseteq R_1[x]$, where $R_1$ is the inseparable closure of $R$—see [1], such that (a) every polynomial $p(x) \in \mathcal{P}$ is normal (in the sense that adjoining one root of $p(x)$ splits $p(x)$ into linear factors) and unambiguous. (b) If $R' \supset R$ splits every polynomial in $\mathcal{P}$ and is generated over $R$ by the roots of these polynomials, then $R' \subseteq K_{CR}$. (c) Each $p(x) \in \mathcal{P}$ is defined and monic on some idempotent $e_p(eR)$ and $p(x) = p(x)e_p$. (d) If $A \subseteq \mathcal{P}$ is countable, there is a countable $B$ with $A \subseteq B \subseteq \mathcal{P}$ such that if $R'$ results from $R$ by adjoining roots of all the polynomials $p(x) \in B$ (in the sequentially prime way—see [1]), then in $R'[x]$ every polynomial $p(x) \in \mathcal{P}$ factors on $e_p$ into unambiguous monic factors. We shall call such a $\mathcal{P}$ a thin basis for $R$.

We shall show that if $R$ is prime over $R$, then $\overline{R}$ is sequentially prime over $R$ and consequently that $R$ has a prime integral closure if and only if $R$ is thin.

**Remark.** In definition (5) above the only important conditions are (b) and (d); i.e. if we have a set of polynomials which satisfies (b) and (d), then we can construct a set satisfying (a)–(d). Notice also that if $R$ is thin, then $R \supset T$.

From now on, when $R \subseteq T$, we shall assume that $R$ is inseparably closed (i.e. every purely inseparable polynomial in $R[x]$ has a root). This involves no loss of generality since the inseparable closure $R_1$ of $R$ always exists and is prime and in fact sequentially prime over $R$. If $R$ is inseparably closed instead of unambiguous polynomials, we can talk of irreducible polynomials (see [1]). Also all irreducible polynomials are then separable and, consequently, we have the primitive element theorem holding.

Let $R \subseteq T$ and let $\mathcal{P} = \{ p(x) \in R[x] | p(x) \text{ is normal, monic and unambiguous} \}$. Let

$$R^* = \prod_{j \in J} R[[x_p | p \in \mathcal{P}]],$$

where the product is over all isomorphism types of $R[[x_p | p \in \mathcal{P}]]$ such that $p(x_p) = 0$ for all $p \in \mathcal{P}$. Let $\overline{R}$ be the subring of $R^*$ generated by the sequences $x_p = (x_{p,j})_{j \in J}$, over $R$. It follows from Lemma 1 of [2] or Lemma 2 of [1] that $\overline{R}$ is a commutative regular ring. It is not hard to see that $\overline{R} \subseteq K_{CR}$ ($\overline{R}$ is algebraically closed at each point of $S_{\overline{R}} = \text{Spec}(\overline{R})$ and since $S_{\overline{R}}$ is compact, $\overline{R}$ is integrally closed). $\overline{R}$ is a free closure of $R$ in the sense that if $R \subseteq R_1 \subseteq K_{CR}$, then there is a homomorphism of $\overline{R}$ into $R_1$ over $R$—in fact one of the projections will do.

Suppose that $R$ has a prime integral closure $\overline{R}$. Let $\nu: \overline{R} \to \overline{R}$ be a fixed embedding over $R$. For each $\beta \in \overline{R}$ there is a finite set $X_\beta \subseteq \{ x_p | p \in \mathcal{P} \} \subseteq \overline{R}$ such that $\nu(\beta) \subseteq R[x_p | x_p \in X_\beta]$. If $A \subseteq \overline{R}$, define $A' \subseteq \overline{R}$ as follows: $A_0 = A$, $A_{i+1} = \{ \text{all roots in } \overline{R} \text{ of polynomials } p(x) \in R[x] \text{ such that } x_p \in \bigcup \beta \in A_i X_\beta \}$ and $A' = \bigcup \omega A_i$. Notice that if $p(x) \in R[x]$, then all the roots of $p(x)$ in $\overline{R}$ are generated by a finite number of roots over $R$, since $S_{\overline{R}} = S_R$. It follows that if $A \subseteq \mathfrak{M}_0$, then $R[A']$ is countably generated over $R$ and, in fact, if $A \subseteq B$ with $B - A$ countable, then $R[B']$ is countably generated over $R[A']$.

Let $\overline{R} = \{ x_\alpha | \alpha < \lambda \}$ where each $x_\alpha$ is a root of a polynomial $p(x) \in \mathcal{P}$. Define $A_\alpha = \{ \{ x_\alpha | \alpha \leq \alpha \} \}$. If $R = R[A'] \subseteq \overline{R}$ and $\overline{R}_\alpha = \{ x_p \in \overline{R} | a \text{ is a root of } p(x) \text{ for some } a \in \overline{R}_\alpha \}$.
It is clear that $R_\alpha = \psi^{-1}(\bar{R}_\alpha)$, that $R_\delta = \bigcup_{\alpha \leq \delta} R_\alpha$ for limit ordinals $\delta \leq \lambda$, that $\bar{R}_\lambda = R$ and that $R_{\alpha+1}$ is countably generated over $R_\alpha$.

**Lemma 1.** (i) $R_\alpha$ is prime over $R$.
(ii) $R_{\alpha+1}$ is prime over $R_\alpha$.
(iii) $R_{\alpha+1}$ is sequentially prime over $R_\alpha$.

**Proof.** (i) is trivial.
(ii) Since $R_\alpha$ is free over $R$ there is a projection $\mu: \bar{R}_\alpha \rightarrow R_\alpha$ over $R$. It is easy to see that $\mu \circ \nu$ is an automorphism of $R_\alpha$. Let $\mathcal{S}' = \text{Ker}(\mu) \subset \bar{R}_\alpha$ and let $\mathcal{S} = \mathcal{S}' \bar{R}$. Then since $R$ and $\bar{R}_\alpha$ are models of $K_{CR}$, $\mathcal{S} \cap \bar{R}_\alpha = \mathcal{S}'$. Also $\mu: \bar{R}_\alpha / \mathcal{S}' \rightarrow R_\alpha$ is an isomorphism. It is easy to see that $\bar{R} / \mathcal{S}$ is free over $R_\alpha = \bar{R}_\alpha / \mathcal{S}'$ (in the same sense that $R$ is free over $R$). Let $R_\alpha \subset R_2 \models K_{CR}$. Then there is a homomorphism $\mu_1: \bar{R} / \mathcal{S} \rightarrow R_2$ over $R_\alpha$ so that $\mu_1 \circ \nu: R \rightarrow R_2$ is an embedding. Hence $R$ (and thus $R_{\alpha+1}$) is prime over $R_\alpha$. Hence, since $R_{\alpha+1}$ is countably generated over $R_\alpha$, by the remark following Theorem 2 of [1], $R_{\alpha+1}$ is sequentially prime over $R_\alpha$, and (ii) and (iii) are proved.

**Corollary.** If $\bar{R}$ is a prime integral closure of $R \models K_{CR}$, then $\bar{R}$ is sequentially prime over $R$.

**Proof.** The results of [1] show that $\bar{R} \models T$. The inseparable closure $R_\iota$ of $R$ is always sequentially prime over $R$ and $R_\iota \models T$ so the above construction and Lemma 1 show that $\bar{R}$ is sequentially prime over $R_\iota$.

**Lemma 2.** If $\bar{R}$ is the prime integral closure of $R$, then $R$ is thin.

**Proof.** Let $\bar{R} = \bigcup_{\alpha < \lambda} R_\alpha$ where $R_{\alpha+1} = R_\alpha[\alpha_\iota]$ and $p_\alpha(\alpha_\iota) = 0$ with $p_\alpha(x) \in R_\alpha[x]$ irreducible. Let $p_\alpha^*(x) \in \bar{R}[x]$ be the unique irreducible polynomial in $\bar{R}[x]$ such that $p_\alpha(x)|p_\alpha^*(x)$. Without loss of generality we may assume that $p_\alpha(x)$ and $p_\alpha^*(x)$ are normal (see the proof of Lemma 1). A set $A \subset \bar{R} - R$ is called downwardly closed if: (i) if $a \in R[A]$ and at some point $z \in S_R$ the first time $a(z)$ occurs in the sequence $\bar{R}_\gamma / z$ is at stage $\alpha$ with $a(z)$ being a combination over $R$ of $a_{1\iota}, \ldots, a_{n\iota}$ say, then $a_{1\iota} \in R[A]$ for $i = 1, \ldots, n$; and (ii) $A = A'$. In the proof of Lemma 1 we saw that if $A = A'$, then $R[A] \models T$, so if $A$ is downwardly closed then $R[A] \models T$. For each downwardly closed $A \subset \bar{R}$ let $Y_A^\alpha$ be a factoring of $p_\alpha^*$ into irreducible factors in $R[A]$. Call two such factorings $Y_A^\alpha$ and $Y_A^{\alpha'}$ essentially different if at some point $z \in S_R$ they are different. We now claim that for fixed $\alpha$ there are only finitely many essentially different $Y_A^\alpha$'s (with $A$ downwardly closed). This follows from the fact that $p_\alpha^*$, factors only finitely often in the well-ordered sequence $\bar{R}_\gamma$ since each $R_\gamma \models T$, and that $S_R$ is compact. We leave the details to the reader. For each $\alpha$ choose idempotents $e_{\alpha,i}, i = 1, \ldots, n_\alpha$ such that for any downwardly closed $A$ each $p_\alpha^*(x)e_{\alpha,i}$ factors into monic irreducible factors on $e_{\alpha,i}$ for each $i$. Let

$$\mathcal{P} = \{ p_\alpha^*(x)e_{\alpha,i} | \alpha < \lambda, i = 1, \ldots, n_\alpha \}.$$ 

Certainly $\mathcal{P}$ satisfies all the conditions of definition (5) except perhaps (d). Let $A \subset \mathcal{P}$ with $A = S_0$ and let $B_1$ be the downward closure of $A$ (defined as follows: For each $a \in R[A] - R$ and each $z \in S_R$, adjoin to $A$ the elements
$a_{\gamma_1}, \ldots, a_{\gamma_n}$ defined above. Since $S_R$ is compact, this will only involve considering a finite number of $z$'s. Call the new set $D$. Let $A_1 = D'$. Obtain $A_{i+1}$ from $A_i$ in the same way that $A_1$ was obtained from $A$. The downward closure of $A$ is $\bigcup_{i<\omega} A_i$). We must show that there is a countable subset $B \subset B_1$ such that adjoining roots for all polynomials in $B$ (in the prime way) causes every polynomial in $B_1$ to split. Since $R$ is separable over $R$ for each $\{a_1, \ldots, a_n\} \subset R$ there are essentially only finitely many regular rings between $R$ and $R[a_1, \ldots, a_n]$. By this we mean that there is a finite set of regular rings $R_j, j = 1, \ldots, k$, with $R_j \subset R[a_1, \ldots, a_n]$ and $R_j$ finitely generated over $R$ such that at each point $z \in S_R$, if $R_z$ denotes the field (i.e. stalk) above $z$, then all the fields between $R_z$ and $R[a_1, \ldots, a_n]$ occur among the $R_j$. For each $\{a_1, \ldots, a_n\} \subset A$ we can look at the rings $R_j$ defined as above and choose a finite set of generators $A_j$ for $R_j$ over $R$. Let $\bar{A}$ be the union of all these $A_j$ for all finite subsets $\{a_1, \ldots, a_n\}$ of $A$. Then $\bar{A}$ is countable and in obtaining $D$ from $A$ as above instead of considering all elements of $R[A] - R$ we need only consider all elements of $\bar{A}$. Call this set $\bar{D}$. Let $A_1 = \bar{D}'$ etc. and $B = \bigcup_{i<\omega} A_i$. Then $B$ is countable and downwardly closed. In fact $R[B] = R$ [downward closure of $A$]. From the definition of $\mathfrak{p}$ it is clear that $B$ has the required properties.

**Lemma 3.** If $R \subset R_1 \subset R_2$ with $R_1 \not\cong T$ and $R_j (j = 1, 2)$ prime over $R$ then $R_2$ is prime over $R_1$.

**Proof.** Let $\bar{R}_1$ be constructed from $R_1$ as above. We then have

$$
\begin{array}{ccc}
R & \subset & R_1 \\
\subset & \phi & \subset \\
\cap & \cap & \cap \\
\downarrow & \downarrow & \downarrow \\
\bar{R}_1 & \bar{R}_2
\end{array}
$$

Proof. Let $\mathfrak{p}$ be a thin basis for $R$. Let $A \subset \mathfrak{p}$. Then there is a $B$ ($A \subset B \subset \mathfrak{p}$) with $\bar{A} + \mathfrak{N}_0 = \bar{B} + \mathfrak{N}_0$ so that every $p \in \mathfrak{p}$ factors in $R_B$ (obtained by adjoining roots of polynomials in $B$) into the product of irreducible monic factors on $e_p$.

We prove by induction on $A$ that if $A \subset \mathfrak{p}$, then there is a sequentially prime extension $R_A$ of $R$ which splits every polynomial in $A$ and with $R_A \not\cong T$, and such that in $R_A[x]$ every polynomial $p \in \mathfrak{p}$ factors into the product of monic irreducible factors on $e_p$. If $A$ is countable this is trivial. Suppose the assertion is true for all cardinals $< \bar{A}$. Let $B$ correspond to $A$ as above. Write $A = \bigcup_{\alpha<\lambda} A_\alpha$ with $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ for limit ordinals $\delta \leq \lambda$, $A_{\alpha+1} \subset A_\alpha$ and $\bar{A}_\alpha < \bar{A}$ for all $\alpha < \lambda$. Let $B = \bigcup_{\alpha<\lambda} B_\alpha$ with $B_\alpha$ corresponding to $A_\alpha$ as above. Then by induction $R_{B_\alpha}$ exists for each $\alpha < \lambda$. It is clear that $R_{B_\alpha} \not\cong T$ (since in $R_{B_\alpha}$ every polynomial in $\mathfrak{p}$ factors into a product of monic irreducible factors) for each $\alpha < \lambda$. Thus by Lemma 3, $R_{B_{\alpha+1}}$ is prime over $R_{B_\alpha}$ and hence $R_B = \bigcup_{\alpha<\lambda} R_{B_\alpha}$ is prime over $R$.

From Corollary 1 and Lemma 4 we immediately get the

**Theorem.** If $R \not\cong K_{CR}$ then $R$ has a prime integral closure if and only if $R$ is thin.
where \( \varphi \) is an embedding of \( R_2 \) into \( \overline{R}_1 \) over \( R \) which exists because \( R_2 \) is prime over \( R \). This diagram need not commute, but we do have \( \nu(r) = \varphi(r) \) for \( r \in R \). We shall show that there exists an automorphism \( \psi \) of \( \overline{R}_1 \) over \( R \) such that the above diagram with \( \varphi \) replaced by \( \psi^{-1} \circ \varphi \) does commute. The lemma then follows from the freeness properties of \( \overline{R}_1 \) over \( R_1 \).

\( \overline{R}_1 \) is generated by the \( x_p, p \in \mathcal{P}, \) over \( R_1 \). For \( a \in R_1 \) let \( a_i, i = 1, \ldots, n_a, \) denote the conjugates of \( a \) over \( R \), and for \( p(x) \in \mathcal{P} \) let \( p_i(x), i = 1, \ldots, n_p, \) denote the conjugates of \( p(x) \) over \( R \). Notice that if \( p(x) \in \mathcal{P} \), then \( p_i(x) \in \mathcal{P} \), and since \( R_1 \models T, (p_i(x), p_j(x)) = 1 \) for \( i \neq j \).

For \( a \in R_1 \) we have \( \varphi(a) = \sum a_i e_i \) where the \( e_i \) are disjoint idempotents of \( \overline{R}_1 \) and \( \sum e_i = 1 \). Similarly we have \( \varphi(p(x)) = \sum_{i=1}^{n_p} p_i(x)e_i. \)

Define \( \psi: \overline{R}_1 \to \overline{R}_1 \) as follows:

\[
\psi(a) = \varphi \circ \nu^{-1}(a) \quad \text{for} \quad a \in \nu(R_1),
\]

\[
\psi(x_p) = \sum_{i=1}^{n_p} x_{p_i} e_i.
\]

It is obvious that \( \psi \) is a homomorphism because of the freeness properties of \( \overline{R}_1 \) over \( R_1 \). \( \psi \) is locally one-to-one (i.e. on each stalk) and hence one-to-one. Also \( x_p \in \text{Range}(\psi) \) so \( \psi \) is onto. Therefore \( \psi \) is an automorphism with the required properties.

**Lemma 4.** If \( R \) is thin, then \( R \) has a prime integral closure.

**Remark.** The condition that \( R \) be thin is not a first order condition since every countable model of \( T \) is thin. Hence for \( R \) uncountable the necessary and sufficient condition for \( R \) to have a prime integral closure is not first order, while for countable \( R \) it is.

**References**


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