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CHARACTERIZING $\mathcal{C}_3$ (THE LARGEST COUNTABLE Π_3^1 SET)

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ABSTRACT. Assume projective determinacy. For any real α, let $\lambda_3^\alpha = \sup\{\xi | \xi$ is the type of a prewellordering of the reals which is Δ_3^1 in $\alpha\}$. Then, $\mathcal{C}_3$, the largest countable Π_3^1 set of reals, is equal to $\{\alpha | \forall\beta(\lambda_3^\alpha \leqslant \lambda_3^\beta \Rightarrow \alpha$ is Δ_3^1 in $\beta)\}$. This result, which is true for all odd levels and generalizes a previously known characterization of $\mathcal{C}_1$, answers a question of Kechris.

Let E_n be Π_n^1 if n is odd, and Σ_n^1 if n is even. It is an important result of recent descriptive set theory that, assuming projective determinacy ("PD"), there is for each n a maximum countable E_n set of reals. Call it $\mathcal{C}_n$. (As usual, a real is a function from ω to ω.) At level one a form of this result is provable from ZF alone (namely that there exists a Π_1^1 set maximum with respect to the property "containing no perfect subset", from now on abbreviated "thin"). $\mathcal{C}_1$ can be characterized as

$$(*) \qquad \{\alpha | \forall\beta(\omega_1^\alpha \leqslant \omega_1^\beta \Rightarrow \alpha \text{ is } \Delta_1^1 \text{ in } \beta)\}.$$

There are two obvious ways to generalize "$(*) = \mathcal{C}_1$" to higher odd levels. Martin and Solovay have shown that one such generalization is false. We show the other one true.

For unexplained notation and terminology see [Kechris, 1]. In particular, we use α, β, γ for reals and η, ξ for ordinals. For present purposes 3 is a typical odd integer greater than 1.

In the expression $(*)$ there are two ways to interpret "ω_1^α": "the least ordinal not the type of a well ordering of integers which is Δ_1^1 in α" or "the sup of all Δ_1^1-in-α prewellorderings of $\mathbf{R}$ (the set of reals)."

Let

$$\delta_3^1(\alpha) = \sup\{\xi | \xi \text{ is the type of a} \Delta_3^1(\alpha) \text{ wellordering in } \omega\},$$

$$\lambda_3^\alpha = \sup\{\xi | \xi \text{ is the type of a } \Delta_3^1(\alpha) \text{ prewellordering of } \mathbf{R}\},$$

$$\alpha \leqslant {}_3\beta \Leftrightarrow \alpha \text{ is } \Delta_3^1 \text{ in } \beta.$$

Then the obvious liftings of $(*)$ are:

$$\Delta_3 = \{\alpha | \forall\beta(\delta_3^1(\alpha) \leqslant \delta_3^1(\beta) \Rightarrow \alpha \leqslant {}_3\beta)\},$$

$$\Lambda_3 = \{\alpha | \forall\beta(\lambda_3^\alpha \leqslant \lambda_3^\beta \Rightarrow \alpha \leqslant {}_3\beta)\}.$$

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[Kechris, 2] contains a proof that $\Delta_3 = Q_3$, a fact due to Martin and Solovay. We will show that $\Lambda_3 = \mathcal{C}_3$. ($Q_3$ is defined in [Kechris, 1]. It is a proper subset of $\mathcal{C}_3$, proper superset of the collection of Δ^1_3 reals, and in many respects analogous to the set of hyperarithmetic reals.)

From now on assume PD. With the exception of Theorem 1(ii) our results follow from $\mathbf{\Delta}^1_2$ determinacy (1(ii) uses $\mathbf{\Delta}^1_4$), for $\mathbf{\Delta}^1{}_2$ determinacy suffices to prove the following facts: (Moschovakis) Every Π^1_3 set admits a Π^1_3 norm. (Kechris) $\mathcal{C}_3$ is the largest thin Π^1_3 set. (Mycielski and Swierczkowski) $\mathbf{\Pi}^1_3$ sets are Lebesgue measurable. (Martin) If A is Σ^1_3 in α and contains a real which is not $\leqslant {}_3\alpha$, then A is thick (i.e., not thin); in the immediate future we will refer to this fact simply as "'Martin's theorem'".

Fix $\mathcal{P} \subseteq \omega \times \mathbf{R}$, a Π^1_3 set which ω-parametrizes the Π^1_3 subsets of $\mathbf{R}$ and is such that for each α, $\{i|(i,\alpha) \in \mathcal{P}\}$ is a complete $\Pi^1_3(\alpha)$ subset of ω. Let φ be a Π^1_3 norm on $\mathcal{P}$. It is a theorem of Moschovakis that for any α, $\lambda^\alpha_3 = \sup\{\varphi(i,\alpha)|(i,\alpha) \in \mathcal{P}\}$. For an extensive list of references for these and related facts, see [Kechris, 1].

DEFINITION. If Ψ is a Π^1_{2n+1} norm on A and $A \subseteq \mathbf{R}$, let

$$\min(\Psi) = \{\alpha|\alpha \in A \text{ and } \forall\beta(\Psi(\beta) = \Psi(\alpha) \Rightarrow \alpha \text{ is } \Delta^1_{2n+1} \text{ in } \beta)\}.$$

Notice that if Ψ is a Π^1_{2n+1} norm, then $\min(\Psi)$ is a Π^1_{2n+1} set.

LEMMA. *Assume PD. Let A be Π^1_{2n+1} and Ψ a Π^1_{2n+1} norm on A. Then,* $\min(\Psi) \subseteq \mathcal{C}_{2n+1}$.

PROOF. For notational simplicity, take $n = 1$. Let $\leqslant^* = \{(\alpha,\beta)|\alpha,\beta \in \min(\Psi) \text{ and } \Psi(\alpha) \leqslant \Psi(\beta)\}$. Then $\leqslant^*$ is a Π^1_3 prewellordering each of whose levels is countable. (A level of $\leqslant^*$ is an equivalence class of the relation $\equiv^*$: $\alpha \equiv^* \beta \Leftrightarrow \alpha \leqslant^* \beta$ and $\beta \leqslant^* \alpha$.) Therefore the field of $\leqslant^*$, which is the Π^1_3 set $\min(\Psi)$, contains no perfect subset. This is proved by a standard argument: If B is a perfect subset of $\min(\Psi)$, let F be a recursive (in a code for B) map sending $\mathbf{R} \xrightarrow{1\text{-}1} B$. Let $\alpha \leqslant' \beta \Leftrightarrow F(\alpha) \leqslant^* F(\beta)$. Then $\leqslant'$ is $\mathbf{\Pi}^1_3$. So $\leqslant'$ is a prewellordering whose field is $\mathbf{R}$, each of whose levels is countable; and $\leqslant'$ is Lebesgue measurable. It is a classical fact that this contradicts Fubini's theorem. □

DEFINITION. Let $n \in \omega$. ξ n-captures $\alpha \Leftrightarrow \forall\beta(\xi < \lambda^\beta_n \Rightarrow \alpha \text{ is } \Delta^1_n \text{ in } \beta)$.

THEOREM 1. *Assume PD. Let $n \in \omega$, $n \geqslant 1$. Then*
(i) $\mathcal{C}_{2n+1} = \{\alpha|\exists\xi(\xi < \lambda^\alpha_n$ *and* ξ *n-captures* $\alpha)\}$,
(ii) $\mathcal{C}_{2n+2} = \{\alpha|\exists\xi(\xi$ *n-captures* $\alpha)\}$.

PROOF. Again take $n = 1$.

(i) Suppose $\alpha \in \mathcal{C}_3$. Let i be such that $\mathcal{C}_3 = \{\beta|(i,\beta) \in \mathcal{P}\}$. Then by 'Martin's Theorem' on Σ^1_{odd} sets, $\varphi(i,\alpha)$ captures α: Given $\varphi(i,\alpha) < \lambda^\beta_3$, choose j such that $\varphi(j,\beta) \geqslant \varphi(i,\alpha)$; then $\{\gamma|\varphi(i,\gamma) \leqslant \varphi(j,\beta)\}$ is Δ^1_3 in β and thin, but contains α. Suppose conversely, that $\xi < \lambda^\alpha_3$ and ξ captures α. Let j be such that $(j,\alpha) \in \mathcal{P}$ and $\xi \leqslant \varphi(j,\alpha)$. Let A be $\{\beta|(j,\beta) \in \mathcal{P}\}$ and Ψ be the norm on A induced by φ (i.e., $\Psi(\beta) = \varphi(j,\beta)$). Then $\alpha \in \min(\Psi)$, because $(\beta \in A \text{ and } \Psi(\alpha) = \Psi(\beta)) \Rightarrow \varphi(j,\alpha) = \varphi(j,\beta), \Rightarrow \xi < \lambda^\beta_3, \Rightarrow \alpha \leqslant {}_3\beta$.

(ii) The inclusion from left to right follows from part (i) because $\alpha \in \mathcal{C}_4 \Leftrightarrow \exists\beta \in \mathcal{C}_3(\alpha \leqslant {}_T\beta)$. To prove the other inclusion let $B = \{\alpha|\exists\gamma, i(\forall\beta,$

$j(\varphi(\beta,j) = \varphi(\gamma,i) \Rightarrow \alpha \leqslant {}_3\beta))\}$. Then B is Σ_4^1 and contains every real which is capturable. It will suffice to show that B contains no perfect subset. Let

$$\Psi(\alpha) \simeq \inf\{\varphi(\gamma,i)|\forall\beta,j(\varphi(\beta,j) = \varphi(\gamma,i) \Rightarrow \alpha \leqslant {}_3\beta)\}.$$

Then Ψ is a Σ_5^1 norm on B each of whose levels is countable. Now repeat the argument in the lemma. □

Theorem 1 is a generalization of the following level one fact: Say that α is Δ_1^1 in ξ iff ξ is a countable ordinal and $\forall\beta\,(\beta$ codes $\xi \Rightarrow \alpha$ is Δ_1^1 in $\beta)$. Then $\mathcal{C}_1 = \{\alpha|\exists\xi < \omega_1^\alpha(\alpha \text{ is } \Delta_1^1 \text{ in } \xi)\}$ and $\mathcal{C}_2 = \{\alpha|\exists\xi(\alpha \text{ is } \Delta_1^1 \text{ in } \xi)\}$. For further information about this notion, see [Kechris, 2].

THEOREM 2. *Assume PD. Let $n \in \omega$. Then,*

$$\{\alpha|\forall\beta(\lambda_{2n+1}^\alpha \leqslant \lambda_{2n+1}^\beta \Rightarrow \alpha \textit{ is } \Delta_{2n+1}^1 \textit{ in } \beta)\} = \mathcal{C}_{2n+1}.$$

PROOF. Yet again, let $n = 1$. That $\mathcal{C}_3 \subseteq \Lambda_3$ is immediate from Theorem 1. For the converse suppose that $\alpha \in \Lambda_3$. Let α' be the Δ_3^1 jump of α, i.e., the characteristic function of $\{i|(i,\alpha) \in \mathcal{P}\}$.

Claim. α' is captured by λ_3^α.

PROOF. Suppose that $\lambda_3^\alpha < \lambda_3^\beta$. Then, since $\alpha \in \Lambda_3$, $\alpha \leqslant {}_3\beta$. But then the "Spector criterion" (see [Kechris, 1]) implies that $\alpha' \leqslant {}_3\beta$. This proves the Claim. By Theorem 1, $\alpha' \in \mathcal{C}_3$. But α and α' have the same Q_3 degree, and $\mathcal{C}_3$ is closed under Q_3 equivalence. (See [Kechris, 1].) □

Here is an external characterization of $\mathcal{C}_{2n+1}$ proved along the same lines:

THEOREM 3. *Assume PD. Let B_{2n+1} = the Boolean algebra generated by the Π_{2n+1}^1 subsets of* $\mathbf{R}$. *Then, $\mathcal{C}_{2n+1}$ is the largest countable element of B_{2n+1}.* (*Note*: *if using only $\boldsymbol{\Delta}_{2n+2}^1$ determinacy, "countable" would have to be replaced by "thin".*)

PROOF. Every element of B_3 can be written in disjunctive normal form as $(A_1 \cap B_1) \cup \cdots \cup (A_k \cap B_k)$, with each $A_i \in \Pi_3^1$, each $B_i \in \Sigma_3^1$. It will therefore suffice to show that $A \in \Pi_3^1$, $B \in \Sigma_3^1$, and $A \cap B$ countable, together imply that $A \cap B \subseteq \mathcal{C}_3$. Let A and B be so and Ψ be a Π_3^1 norm on A. Then $A \cap B \subseteq \min(\Psi)$. ('Martin's Theorem' again: Let $\alpha \in A \cap B$ and $\Psi(\beta) = \Psi(\alpha)$. Let $C = \{\gamma \in B|\Psi(\gamma) \leqslant \Psi(\beta)\}$. Then $\alpha \in C$; and since C is a countable set Σ_3^1 in β, every element of C is $\leqslant {}_3\beta$.)

REFERENCES

[Kechris, 1] *The theory of countable analytic sets*, Trans. Amer. Math. Soc. **202** (1975), 259–297.
[Kechris, 2] *Countable ordinals and the analytic hierarchy*, Pacific J. Math. (to appear).

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