ON UNIVALENT POLYNOMIALS

J. R. QUINE

Abstract. We define $V_n \subseteq \mathbb{C}^{n-1}$ to be the set of $(n-1)$-tuples $(a_2, \ldots, a_n)$ such that the polynomial $p(z) = z + a_2z^2 + \cdots + a_nz^n$ is univalent, i.e., one-to-one in $|z| < 1$. In this paper we construct a real polynomial $h$ of degree $4(2(n-1)^2 - 1)(n-1)$ such that if $(a_2, \ldots, a_n)$ is in the boundary of $V_n$ then $h(\text{Re } a_2, \text{Im } a_2, \ldots, \text{Re } a_n, \text{Im } a_n) = 0$. This shows that the boundary of $V_n$ is a subset of an algebraic submanifold of $\mathbb{R}^{2(n-1)}$.

We define $V_n \subseteq \mathbb{C}^{n-1}$ to be the set of $(n-1)$-tuples $(a_2, \ldots, a_n)$ such that the polynomial $p(z) = z + a_2z^2 + \cdots + a_nz^n$ is univalent, i.e., one-to-one in $|z| < 1$. $V_n$ may also be considered as a subset of $\mathbb{R}^{2(n-1)}$. A convenient description of $V_n$ would be as a region bounded by algebraic submanifolds of $\mathbb{R}^{2(n-1)}$. In Brannan [1], however, $V_3$ is described as the intersection of regions of this sort. In this paper we construct a real polynomial $h$ of degree $4(2(n-1)^2 - 1)(n-1)$ such that if $(a_2, \ldots, a_n)$ is in the boundary of $V_n$ then $h(\text{Re } a_2, \text{Im } a_2, \ldots, \text{Re } a_n, \text{Im } a_n) = 0$. This shows that the boundary of $V_n$ is a subset of an algebraic submanifold of $\mathbb{R}^{2(n-1)}$.

In the following, let $p(z) = a_1z + a_2z^2 + \cdots + a_nz^n$, where $a_1 = 1$.

Definition. We say $w$ is a vertex of the curve $p(e^{i\phi})$, $0 < \phi < 2\pi$, if $p(x) = p(y) = w$ for $x \neq y$, $|x| = |y| = 1$. We say $w$ is a cusp of the same curve if $p(x) = w$ and $p'(x) = 0$. An elementary computation gives

$$G(x, y) = \begin{cases} \frac{(p(x) - p(y))/(x - y)}{p'(x)} & \text{when } x \neq y, \\ p'(x) & \text{when } x = y. \end{cases}$$

A vertex or cusp of $p(e^{i\phi})$ corresponds to a pair $(x_0, y_0)$ such that $|x_0| = |y_0| = 1$ and $G(x_0, y_0) = 0$. Elementary computations give

$$G(x, y) = \sum_{k=0}^{n-1} b_k(x)y^k$$

where

Presented to the Society, November 11, 1973; received by the editors December 10, 1973.


Key words and phrases. Univalent, polynomials.
$b_k(x) = \sum_{j=0}^{n-k-1} a_{j+k+1}x^j.$

Now define

$$G^*(x,y) = x^{n-1}y^{n-1}G(l/x, l/y).$$

We have

$$G^*(x,y) = \sum_{k=0}^{n-1} C_k(x)y^k$$

where

$$C_k(x) = \sum_{j=0}^{n-1} \bar{a}_{2(n-1)-(j+k)+1}x^j.$$ 

From the definition of $G^*$ we see that if $|x_0| = |y_0| = 1$ then $G(x_0, y_0) = 0 \iff G^*(x_0, y_0) = 0$. Now let $R(x)$ be the resultant of $G(x,y)$ and $G^*(x,y)$ as polynomials in $y$, that is, as polynomials in $\mathbb{C}[x][y]$. (See van der Waerden [4, p. 83].)

**Theorem 2.** The point $(a_2, \ldots, a_n)$ is in the interior of $V_n$ if and only if $R(x) \neq 0$ for $|x| = 1$.

**Proof.** If $w$ is a vertex or a cusp of $\rho^{(\phi)}$ then we can find $x_0$ and $y_0$ such that $|x_0| = |y_0| = 1$, $p(x_0) = w$ and $G(x_0, y_0) = G^*(x_0, y_0) = 0$. From the basic properties of the resultant (see van der Waerden [4]) and by symmetry, $R(x_0) = R(y_0) = 0$. Thus the preimages on $|z| = 1$ of the vertices and cusps of $p(e^{i\phi})$ are among the zeros of $R(x)$. Thus if $R(x) \neq 0$ for $|x| = 1$, then by Theorem 1, $(a_2, \ldots, a_n)$ is in the interior of $V_n$.

Conversely, suppose $R(x_0) = 0$ for $|x_0| = 1$. Then we can find $y_0$ such that $G(x_0, y_0) = G^*(x_0, y_0) = 0$, i.e. $G(x_0, y_0) = G^*(x_0, 1/y_0) = 0$. If $|y_0| = 1$ then $(a_2, \ldots, a_n)$ is not in the interior of $V_n$ by Theorem 1. If $|y_0| \neq 1$ then either $|y_0| < 1$ or $|1/y_0| < 1$. Since $p(x_0) = p(y_0) = p(1/y_0)$, $p$ is not univalent in $|z| < 1$.

Now we investigate $R(x)$ further. The expression for $R(x)$ is a $2(n-1)$ by $2(n-1)$ determinant whose only entries are $0$, $c_k(x)$, and $b_k(x)$, $k = 1, \ldots, n-1$. The formal degrees of $b_k(x)$ and $c_k(x)$ in $x$ are $n-k-1$ and $n-1$ respectively. Thus the term in the determinant expression for $R(x)$ with the highest formal degree is the term

$$(b_0(x))^{n-1}(c_{n-1}(x))^{n-1} = (a_1 + a_2x + \cdots + a_nx^{n-1})^{n-1}(\bar{a}_n + \bar{a}_{n-1}x + \cdots + \bar{a}_1x^{n-1})^{n-1} = (a_n\bar{a}_1)^{n-1}x^{2(n-1)^2} + \cdots + (\bar{a}_n a_1)^{n-1}.$$ 

The formal degree of $R$ in $x$ is $2(n-1)^2$ and this is the actual degree if $a_n \neq 0$.

We see likewise that the degree of $R$ in the variables $a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n$ jointly is $2(n-1)$, if $n > 2$.

We also note that $G(x,y) = G^*(x,y) = 0$ if and only if $G(l/x, l/y)$
\( G^*(1/x, 1/y) = 0 \). Thus \( R(x) = 0 \) if and only if \( R(1/x) = 0 \), that is, the zeros of \( R(x) \) are symmetric in the circle \( |z| = 1 \).

**Theorem 3.** If \( (a_2, \ldots, a_n) \) is on the boundary of \( V_n \), and \( p(x_0), |x_0| = 1 \), is a vertex or a cusp of \( p(e^{i\phi}) \), then \( x_0 \) is a double zero of \( R(x) \).

**Proof.** Consider \( R \) as a function of \( (a_2, \ldots, a_n) \) and of \( x \). Take \( (a_2, \ldots, a_n) \) in the interior of \( V_n \) near \( (a_2, \ldots, a_n) \). Then \( R(a_2, \ldots, a_n, x) \) has a zero \( x_1 \) near \( x_0 \) and \( |x_1| \neq 1 \) by Theorem 2. Since \( 1/x_1 \) is also a zero, \( R(a_2, \ldots, a_n, x) \) has two zeros near \( x_0 \). Therefore \( x_0 \) is a double zero of \( R(a_2, \ldots, a_n, x) \).

Now let \( N = 2(n - 1)^2, S(x) = R'(x), S^*(x) = xN^{-1}S(1/x) \). Let \( Q = Q(a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n) \) be the resultant of \( S \) and \( S^* \). The determinant expression for \( Q \) will be a \( 2(N - 1) \) by \( 2(N - 1) \) determinant whose only entries are 0 and the coefficients of \( S \) and \( S^* \). If \( n > 2 \), each of these coefficients is of degree \( 2(n - 1) \) in \( (a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n) \), so the degree of \( Q \) is \( 4(N - 1)(n - 1) \) in \( (a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n) \).

**Theorem 4.** If \( (a_2, \ldots, a_n) \) is on the boundary of \( V_n \) then \( Q(a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n) = 0 \).

**Proof.** By Theorem 3, \( R(x) \) had a double zero \( x_0 \), with \( |x_0| = 1 \). Therefore \( R'(x_0) = S(x_0) = S^*(x_0) = 0 \). Thus \( S \) and \( S^* \) have a common root and their resultant is zero.

To show that \( Q = 0 \) defines an algebraic submanifold of \( R^{2n} \), we show that \( Q \) is real so that as a polynomial in \( \Re a_2, \Im a_2, \ldots, \Re a_n, \Im a_n, Q \) will have real coefficients. Let \( S(x) = AxN^{-1} + \cdots + B \). Then \( S^*(x) = BxN^{-1} + \cdots + A \). Without loss of generality, assume that \( a_n \neq 0 \) so that \( A = N(a_n^4)^{-1} \) is not zero. Let \( x_j, j = 1, \ldots, N - 1 \), be the zeros of \( S \). Then \( 1/x_j, j = 1, \ldots, N - 1 \), are the zeros of \( S^* \), and

\[
Q = (A/B)^{N-1} \prod_{j,k=1}^{N-1} (x_j - 1/x_k).
\]

See van der Waerden [4, p. 87]. Therefore

\[
Q = |A|^{2(N-1)} \prod_k (1 - |x_k|^2) \prod_{j<k} |1 - x_j/x_k|^2
\]

since \( B/A = \prod_k x_k \). Thus \( Q \) is real.

Summarizing, we have

**Theorem 5.** If \( n > 2 \), the boundary of \( V_n \) is a subset of an algebraic submanifold of \( R^{2(n-1)} \) given by a polynomial of degree \( 4(2(n - 1)^2 - 1)(n - 1) \).

Let us look at the case \( n = 3 \). This has been investigated by Brannan [1]. There is no loss in generality in taking \( a_3 \) real and positive. Thus letting \( a_2 = \xi + i\eta, a_3 = \zeta > 0 \), Brannan showed that \( (a_2, a_3) \in V_3 \) if and only if \( f(\xi, \eta, \zeta, d) \leq 0 \) for \(-1 \leq d \leq 3\) where

\[
f(\xi, \eta, \zeta, d) = (1 + d)(\xi^2/(1 + \xi d)^2 + \eta^2/(1 - \zeta d)^2) - 1.
\]

This shows that the point \( (\xi, \eta) \) belongs to the closed interior of a family of ellipses for fixed \( \zeta \). If \( 0 \leq a_3 \leq 1/5 \), the points on the boundary of \( V_3 \) satisfy
4\{\frac{\xi^2}{(1 + 3\xi)^2} + \frac{\eta^2}{(1 - 3\xi)^2}\} = 1. On the part of the boundary where \(1/5 \leq a_3 \leq 1/3\), a portion of the boundary is in the envelope of the above family of ellipses. Now \(f\) of degree 4 in \(d\) and 4 in \((\xi, \eta, \xi)\) jointly. Using the resultant to eliminate \(d\) from \(f\) and \(\partial f / \partial d\) would give us an equation \(R(\xi, \eta, \xi) = 0\) where \(R\) is at most of degree \((4 + 3)4 = 28\). Our \(Q(a_2, a_3, \alpha)\) is of degree 56. We remark that our method is essentially the same as Brannan's except that Brannan makes the change of variables \(x + y = 2zc, xy = z^2\) and then eliminates \(z\) using Cohn's rule instead of the resultant. The fact that the degree of the boundary under Brannan's computations is half that predicted by Theorem 5 is due to his substituting along the way for \(c^2\). At any rate, it appears that the boundary of \(V_n\) has an equation of high degree even for small \(n\).

REFERENCES


DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306