STRONGLY STRICTLY CYCLIC WEIGHTED SHIFTS

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Abstract. An example is given of a unilateral weighted shift on complex Hilbert space which is strictly cyclic but not strongly strictly cyclic. Similar examples of weighted shifts on the sequence spaces $l^p$, for $1 < p < \infty$, are indicated.

1. Introduction. A unilateral weighted shift operator $T$ on complex separable Hilbert space $H$ is an operator that maps each vector in some orthonormal basis $\{e_n\}_{n=0}^\infty$ into a scalar multiple of the next basis vector: $Te_n = w_n e_{n+1}$ for each $n$. The shift $T$ is said to be strictly cyclic if there exists a vector $x \in H$ such that $A(T)x = H$, where $A(T)$ is the weakly closed algebra generated by $T$. Let $E_n$ denote the span in $H$ of the basis vectors $\{e_i: i \geq n\}$. If the restricted operators $T|E_n$ are all strictly cyclic (for $n = 0, 1, 2, \ldots$), then $T$ is said to be strongly strictly cyclic. A. L. Shields, in his survey of weighted shifts, asked if every strictly cyclic weighted shift is strongly strictly cyclic [3, Question 14]. This note gives an example which shows that, even under the additional hypothesis that the weight sequence of the operator is monotonically decreasing to 0, a strictly cyclic operator $T$ may have a restriction $T|E_1$ which fails to be strictly cyclic.

2. Statement of results. The notation of [3] is followed. Let $T$ denote a unilateral weighted shift on $H$ with a weight sequence $w_n$ such that $w_0 > w_1 > w_2 > \cdots > 0$. Define $\beta(0) = 1$ and $\beta(n) = w_0 w_1 \cdots w_{n-1}$ for $n \geq 1$, so that $w_n = \beta(n+1)/\beta(n)$. Let $\beta(n,j) = (\beta(n)/\beta(j)\beta(n-j))^2$, for $0 \leq j \leq n$, and let $S_\beta(n) = \sum_{j=0}^{n} \beta(n,j)$. In order to determine if $T$ is strictly cyclic it suffices, by the following result, to examine the sequence $\{S_\beta(n)\}$.

Theorem [2, Theorem 3.2], [3, Proposition 32]. If $\{w_n\}$ is monotonically decreasing, then $T$ is strictly cyclic if and only if the sequence $\{S_\beta(n)\}$ is bounded.

Corollary. If $\{w_n\}$ is monotonically decreasing to 0 and if $2 < \limsup S_\beta(n) < \infty$, then $T$ is strictly cyclic but $T|E_1$ is not. Thus $T$ is strictly cyclic but not strongly strictly cyclic.

Proof. It is no loss of generality to assume that $w_0 = 1$: if $w_0 \neq 1$, multiply each weight by $(w_0)^{-1}$ and observe that this change does not affect the values of $\beta(n,j)$ or $S_\beta(n)$. The operator $T|E_1$ is itself a weighted shift, with weights $v_n = w_{n+1}$. Define $\alpha(n)$ for $\{v_n\}$ as $\beta(n)$ was defined for $\{w_n\}$, so that $\alpha(0) = 1 = \beta(1)$ and $\alpha(n) = v_0 v_1 \cdots v_{n-1} = \beta(n+1)$. Thus
\[ a(n,j) = (a(n) / a(j) a(n-j))^2 = (\beta(n+1) / \beta(j+1) \beta(n+1-j))^2. \]

Hence

\[ S_\alpha(n) = \sum_{j=0}^{n} a(n,j) = \sum_{k=1}^{n+1} \left( \frac{\beta(n+1)}{\beta(k) \beta(n+2-k)} \right)^2 + \left( \frac{\beta(n+1)}{\beta(n+2)} \right)^2 \sum_{k=1}^{n+1} \beta(n+2,k) = (w_{n+1})^{-2} [S_\beta(n+2) - 2] \]

since \( \beta(n+2,0) = \beta(n+2,n+2) = 1 \). If \( w_n \downarrow 0 \) and if \( 0 < \lim \sup S_\beta(n) < \infty \), it follows that \( \{ S_\alpha(n) \} \) is unbounded, so that \( T \mid E_1 \) is not strictly cyclic.

3. Construction. To find weights \( \{ w_n \} \) which satisfy the conditions of the Corollary, proceed as follows. Note first that, for \( n \geq 2 \), \( S_\beta(n) = 2 + (w_{n-1})^2 K_n \), where \( K_n \) is a function of \( w_0, \ldots, w_{n-2} \) only. It follows that \( S_\beta(n) \) is an increasing function of \( w_{n-1} \). Consequently, if \( w_0, \ldots, w_{n-2} \) are already chosen, \( w_{n-1} \) may then be chosen to make \( S_\beta(n) \) take on any value larger than \( 2 \). The trick is to achieve large values of \( S_\beta(n) \) with a sequence \( \{ w_n \} \) which is monotonically decreasing to \( 0 \).

Take any value \( L > 2 \), and then choose the \( w_n \) inductively as follows. Let \( w_0 = w_1 = 1 \), and suppose that, for \( n \geq 3 \), the weights \( w_0, \ldots, w_{n-2} \) have been chosen so that \( S_\beta(n-1) \leq L \). Let \( w_{n-1} = w_{n-2} \) if this choice will make \( S_\beta(n) \leq L \) (Case 1); otherwise let \( w_{n-1} = \) the smaller of \( \frac{1}{2} w_{n-2} \) (Case 2a) and that value which would make \( S_\beta(n) = L \) (Case 2b). In either case \( S_\beta(n) \leq L \) and \( w_{n-1} \leq w_{n-2} \).

For integers \( n \) of Case 2, \( S_\beta(n) \geq 3/2 + L/4 > 2 \). For, if Case 2a occurs, then \( 2 + (w_{n-2})^2 K_n > L \). Hence

\[ S_\beta(n) = 2 + (w_{n-2})^2 K_n / 4 \]

\[ = 3/2 + [2 + (w_{n-2})^2 K_n] / 4 > 3/2 + L/4. \]

In Case 2b, \( S_\beta(n) = L > 3/2 + L/4 \). Thus \( \lim \sup S_\beta(n) > 2 \) if Case 2 occurs infinitely often. This must be true, for otherwise the sequence \( \{ w_n \} \) is eventually constant. An easy computation then shows that \( \lim S_\beta(n) = +\infty \), contradicting the fact that \( S_\beta(n) \leq L \). Since \( w_{n-1} \leq \frac{1}{2} w_{n-2} \) for integers \( n \) of Case 2 it follows that the sequence \( \{ w_n \} \) decreases monotonically to \( 0 \).

4. Remarks. The construction above may easily be modified to make \( S_\beta(n) \) unbounded. This yields an example of a weighted shift whose weights decrease monotonically to \( 0 \) and yet which is not strictly cyclic. (A similar example is given by D. A. Herrero [1].)

Weighted shifts may be defined on the sequence spaces \( l^p \). The Theorem of Kerlin and Lambert stated above holds for operators on \( l^p \) with \( 1 < p < \infty \) if it is modified so that \( \beta(n,j) \) is defined by \( \beta(n,j) = (\beta(n) / \beta(j) \beta(n-j))^q \), where \( q = p / (p-1) \). (Kerlin and Lambert prove the Theorem in this general form.) The construction above is easily modified to fit this situation. Thus a strictly cyclic weighted shift on \( l^p \), with \( 1 < p < \infty \), need not be strictly cyclic, even though its weight sequence decreases monotonically to \( 0 \).
REFERENCES


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