PARTITIONS WITH CONGRUENCE CONDITIONS

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Abstract. Let \( A = \bigcup_{i=1}^{q} \{a(i) + \nu M: \nu = 0, 1, 2, \ldots\} \), where \( q, M \) and the \( a(i) \) are positive integers such that \( a(1) < a(2) < \cdots < a(q) < M \). Asymptotic formulae are obtained for \( p(n, k, A) \), \( p^*(n, k, A) \) the number of partitions of \( n \) into \( k \) parts, \( k \) unequal parts respectively, where all the parts belong to \( A \).

Since the celebrated Hardy-Ramanujan paper [4] which gave an asymptotic formula for \( p(n) \), the number of unrestricted partitions of the positive integer \( n \), many authors have evaluated different cases of \( p(n, A) \), the number of partitions of \( n \) into parts belonging to

\[
A = \bigcup_{i=1}^{q} \{a(i) + \nu M: \nu = 0, 1, 2, \ldots\},
\]

where \( q, M \) and the \( a(i) \) are positive integers such that \( a(1) < a(2) < \cdots < a(q) \leq M \). The Hardy-Ramanujan circle method as modified by Rademacher [9] was employed to obtain convergent series expressions for \( p(n, A) \). When \( q = 2, a(2) = M - a(1) \), Niven [8] investigated the case \( M = 6 \), Lehner [6] the case \( M = 5 \) and Livingood [7] the case where \( M \) is any prime \( > 3 \). Later Iseki [5] evaluated \( p(n, A) \) when \( M \) is any composite \( > 3 \) and \( a(1), M \) are relatively prime. Then Hagis [3] considered the case of all odd primes \( M \), where \( q = 2r \) and \( a(h) + a(2r - h) = M \) for \( 1 \leq h \leq r \).

In all these cases, \( A \) is symmetric in the sense that \( a(h) \in A \) implies that \( M - a(h) \in A \) and this ensures that the generating function of the \( p(n, A) \) is a modular form. Rademacher's method then leads to a convergent series representation of \( p(n, A) \). Grosswald [2] investigated the case where \( M \) is any odd prime and \( A \) is an arbitrary asymmetrical set. Then the above method cannot be applied and only asymptotic results are obtained.

In [1], Erdös and Lehner by means of more elementary methods investigated \( p(n, k) \), the number of partitions of \( n \) into \( k \) parts, and proved that \( p(n, k) \sim n^{k-1}/k!(k - 1)! \) as \( n \to \infty \), provided that \( k = o(n^{1/3}) \). This formula does not appear to have been generalized in the above manner and it is the object of this note to do so.

We denote by \( p(n, k, A) \), \( p^*(n, k, A) \) the number of partitions of \( n \) into \( k \) parts, \( k \) unequal parts respectively, where all the parts belong to \( A \). For every
positive integer \( n \), we define \( r \) by \( r = n \) (mod \( M \)), \( 1 \leq r \leq n \). We write

\[
P(x, k, A) = \left\{ \sum_{i=1}^{q} x^{a(i)} \right\}^k = \sum_{r=1}^{kM} c_r x^r,
\]

\[
S_r(P(x, k, A)) = \sum_{r=0}^{k-1} c_r M + r
\]

and \( d, \delta \) for the greatest common divisors \((a(1), \ldots, a(q), M), (a(2) - a(1), \ldots, a(q) - a(1), M)\). \( \delta = M \) for \( q = 1 \). Clearly \( d = (a(1), \delta) \). We can now state our result.

**Theorem.** For any given \( k, r \) for which \( S_r(P(x, k, A)) \neq 0 \),

\[
p(Mn + r, k, A) \sim p^*(Mn + r, k, A) \sim n^{k-1} S_r(P(x, k, A))/k!(k - 1)!
\]

as \( n \to \infty \). If \( \delta \) divides \( r \), \( n \to \infty \) and \( k \to \infty \) through multiples of \( \delta/d \) subject to the condition that \( k = o(n^{1/4}) \), then

\[
p(Mn + r, k, A) \sim p^*(Mn + r, k, A) \sim n^{k-1} q^k \delta/M[k!(k - 1)!].
\]

We observe that the second part of our theorem is established only for \( k = o(n^{1/4}) \) instead of under the condition \( k = o(n^{1/3}) \) of Erdös and Lehner. The proof here is, in fact, quite different from that of [1] where the condition on \( k \) is obtained by using the fact that the number of partitions of \( n \) into \( k \) unequal parts is \( p(n - \frac{1}{2}k(k - 1), k) \). This appears to have no obvious generalization for partitions into parts belonging to \( A \).

**Proof of the Theorem.** We write

\[
G(y) = G(y, x, A) = \prod_{r=0}^{\infty} \prod_{i=1}^{q} \frac{1 - x^{rM + a(i)} y}{1 - x^{rM + a(i)} y}
\]

\[
= 1 + \sum_{k=1}^{\infty} g(k) y^k,
\]

where \( g(k) = g(k, x, A) \) is the generating function of \( p(n, k, A) \). Therefore,

\[
\log G(y) = - \sum_{r=0}^{\infty} \sum_{i=1}^{q} \log(1 - x^{rM + a(i)} y)
\]

\[
= \sum_{r=0}^{\infty} \sum_{i=1}^{q} \sum_{h=1}^{\infty} h^{-1} x^{h(rM + a(i))} y^h
\]

\[
= \sum_{h=1}^{\infty} h^{-1} y^h \beta(h),
\]

where

\[
\beta(h) = \beta(h, x, A) = (1 - x^{hM})^{-1} \sum_{i=1}^{q} x^{ha(i)}.
\]

Hence
\[ G(y) = \exp \left\{ \sum_{h=1}^{\infty} y^h \beta(h) \right\} \]

and so,

\[ g(k) = \sum_{(k)} \prod_{m} \{h(m)\}^{-1} \{m^{-1} \beta(m)\}^{h(m)}, \]

where the sum is taken over all unrestricted partitions of \( k \) of the form \( k = \sum_{m=1}^{h} h(m)m \) and the product is taken over all the different parts \( m \) of the partition.

The term on the right-hand side of (4) corresponding to the partition of \( k \) into \( k \) units is

\[ \frac{(k!)^{-1}(1 - x^M)^{-k} \left\{ \sum_{i=1}^{q} x^{a(i)} \right\}^k}{m!} \]

by (1), and the coefficient of \( x^{Mn+r} \) in this term is

\[ (k!)^{-1} \sum_{\nu=0}^{k-1} c_{\nu M+r} \binom{n - \nu + k - 1}{k - 1}. \]

It follows that, as \( n \to \infty \), this coefficient

\[ \sim n^{k-1} \sum_{\nu=0}^{k-1} c_{\nu M+r}/k! (k - 1)! \]

by (2).

From (3), the general term on the right-hand side of (4) is

\[ \prod_{m} \{h(m)\}^{-1} m^{-h(m)} (1 - x^{M})^{-h(m)} \left\{ \sum_{i=1}^{q} x^{ma(i)} \right\}^{h(m)} \]

Since, for all \( \nu > 0 \), the coefficient of \( x^{\nu} \) in \( \sum_{i=1}^{q} x^{a(i)} \) is not greater than the coefficient of \( x^{\nu} \) in \( \left\{ \sum_{i=1}^{q} x^{a(i)} \right\}^{m} \), the coefficient of \( x^{Mn+r} \) in this general term is less than

\[ C n^{-1} \prod_{m} nm^{-h(m)} (n/m)^{h(m)-1} S_r\{P(k,A)\}/h(m)! \{h(m) - 1\}!, \]

where \( C \) is independent of \( n, k \). Hence, in order to show that

\[ p(Mn + r, k, A) \sim n^{k-1} S_r\{P(x, k, A)\}/k! (k - 1)! \]

as \( n \to \infty \), we must prove that

\[ n^{-k} k! (k - 1)! \sum_{(k)} \prod_{m} n^{h(m)} m^{-2h(m)} /h(m)! \{h(m) - 1\}! = o(1) \]
as \( n \to \infty \), where the sum is taken over all partitions \( \sum h(m)m \) of \( k \) into less than \( k \) parts.

Now, since any partition of \( k \) into \( k - \mu \) parts, where \( \mu < \frac{1}{2}k \), must contain at least \( k - 2\mu \) units, we have

\[
\prod_m h(m)! \{h(m) - 1\}! \geq \Lambda(k - 2\mu)
\]

where \( \Lambda(k - 2\mu) = (k - 2\mu)!/(k - 2\mu - 1)! \) for \( \mu < \frac{1}{2}k \) and \( \Lambda(k - 2\mu) = 1 \) for \( \mu > \frac{1}{2}k \). Also, the number of partitions of \( k \) into \( k - \mu \) parts is less than or equal to \( p(\mu) \), the number of unrestricted partitions of \( \mu \), according as \( \mu > \frac{1}{2}k \) or \( \mu < \frac{1}{2}k \). Therefore, since for all \( \mu > 0 \), \( p(\mu) < \exp\{\pi\sqrt{2\mu/3}\} \), the left-hand side of (6) is less than

\[
\sum_{\mu=1}^{k-1} \exp\{\pi\sqrt{2\mu/3}\} k^{4\mu} n^{-\mu}
\]

\[
< \sum_{\mu=1}^{k-1} \exp\{\pi\sqrt{2\mu/3} + 4\mu \log k - \mu \log n\}
\]

\[
< C' \sum_{\mu=1}^{\infty} \exp\{-\frac{1}{2}\mu \log (n/k^4)\},
\]

where \( C' \) is independent of \( n, k \). (6) follows immediately.

If we write

\[
G^*(y) = G^*(y, x, A) = \prod_{r=0}^{\infty} \prod_{i=1}^{q} \{1 + x^{rM+\sigma(i)} y\}
\]

\[
= 1 + \sum_{k=1}^{\infty} g^*(k) y^k,
\]

then it follows that

\[
\log G^*(y) = \sum_{h=1}^{\infty} (-1)^{h-1} h^{-1} y^h \beta(h)
\]

and

(7) \[
\]

(7)

\[
g^*(k) = \prod_{\mu} h(m)! \{h(m) - 1\}^{-1} (-1)^{m-1} m^{-1} \beta(m) h(m),
\]

The proof of the asymptotic formula for \( p^*(Mn + r, k, A) \) then follows from (7) exactly as the proof of the formula for \( p(Mn + r, k, A) \) followed from (4).

In order to prove the second part of the theorem, we let \( \omega \) be a primitive \( M \)th root of unity. Then, from (1) and (2),

\[
P(\omega^h, k, A) = \sum_{r=1}^{kM} c_r \omega^{rh} = \sum_{r=1}^{M} S_r \omega^{rh}
\]

for \( 1 \leq h \leq M \), where \( S_r = S_r(\{P(\omega^h, k, A)\}) \). It is a simple application of alternant theory to solve these equations and obtain
\begin{equation}
S_r = M^{-1} \sum_{p=0}^{M-1} \omega^{-rp} P(\omega^r, k, A)
\end{equation}

for \(1 \leq r \leq M\). From the definition of \(\delta\), \(\sum_{i=1}^{q} \omega^{rad(i)} = q \omega^{rad(1)}\) whenever \(M/\delta\) divides \(v\) and \(\left| \sum_{i=1}^{q} \omega^{rad(i)} \right| < q\) otherwise. Hence, since \(\delta\) divides \(r\) and \((a(1),\delta) = d\), we see from (1) and (8) that \(S_r \sim q^k \delta/M\) as \(k \to \infty\) through multiples of \(\delta/d\). The second part of the theorem now follows from (5) by letting \(n, k \to \infty\), since (6) follows as before for \(k = o(n^{1/4})\).

REFERENCES


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