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A NAGUMO CONDITION FOR ORDINARY DIFFERENTIAL EQUATIONS[1]

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ABSTRACT. A condition is given which guarantees that each solution of $y^{(n)} = f(x,y,y',\ldots,y^{(n-1)})$ either extends or becomes unbounded on its maximal interval of existence.

Introduction. The classical Nagumo condition for the second order ordinary differential equation

$$y'' = f(x,y,y') \tag{1}$$

is a growth condition on $f(x,y,y')$ which implies that solutions of (1) either extend or become unbounded on their maximal intervals of existence. One formulation of the condition is contained in the following theorem.

THEOREM 1. *Assume that* (1) *is a scalar equation with* $f(x,y,y')$ *continuous on* $(a,b) \times R^2$. *If for each* $M > 0$ *and each compact interval* $[c,d] \subset (a,b)$ *there is a corresponding positive continuous function* $\phi(s)$ *on* $[0,\infty)$ *such that* $|f(x,y,y')| \leqslant \phi(|y'|)$ *for all* (x,y,y') *satisfying* $c \leqslant x \leqslant d$, $|y| \leqslant M$ *and such that* $\int_0^\infty s/\phi(s)ds = +\infty$, *then each solution of* (1) *either extends to* (a,b) *or becomes unbounded on its maximal interval of existence.*

Other formulations from which Theorem 1 follows may be found in [1, p. 428] and [2, p. 353].

This property of solutions which is stated as the conclusion in Theorem 1 along with the assumed existence of solutions of certain types of differential inequalities plays an important role in demonstrating the existence of solutions of boundary value problems not only for second order equations but for higher order equations as well; see for example, [2]–[6].

In this paper we establish a "Nagumo condition" for equations

$$y^{(n)} = f(x,y,y',\ldots,y^{(n-1)}) \tag{2}$$

of arbitrary order n.

1. **Preliminary results.** For a function $g \in C^{(n)}[\alpha,\beta]$ let $M_k = \operatorname{Max}\{|g^{(k)}(x)|: \alpha \leqslant x \leqslant \beta\}$, $0 \leqslant k \leqslant n$. The proof of the following lemma may be found in [7].

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LEMMA 2. *Let M_k, $0 \leqslant k \leqslant n$, be as above. Then for each integer k, $0 < k < n$,*

$$M_k \leqslant C_{nk} M_0^{1-k/n} (M'_n)^{k/n}$$

where

$$C_{nk} = 4e^{2k} n^k k^{-k} \quad \textit{and} \quad M'_n = \operatorname{Max}\{M_n, 2^n n!\, M_0(\beta - \alpha)^{-n}\}.$$

Note. The factor 2^n appearing in the definition of M'_n is not present in the result as stated in [7], however, the author believes that it should be included.

DEFINITION 3. For $p \geqslant 0$ let the function $\phi_p(x)$ be defined by

$$\phi_p(x) = \begin{cases} 1 & \text{for } |x| \leqslant 1, \\ |x|^p & \text{for } |x| \geqslant 1. \end{cases}$$

LEMMA 4. *Assume that the numbers p_i, $1 \leqslant i \leqslant n - 1$, are fixed nonnegative numbers such that $\sum_{i=1}^{n-1} ip_i < n$. Then*

$$\sum_{i=1}^{n-1} |x_i|^{n/i} \Big/ \prod_{i=1}^{n-1} \phi_{p_i}(x_i) \to +\infty \quad \textit{as} \sum_{i=1}^{n-1} |x_i|^{n/i} \to +\infty.$$

PROOF. Assume that $\sum_{i=1}^{n-1} |x_i|^{n/i} = h > 1$. Then $|x_i| \leqslant h^{i/n}$ for each $1 \leqslant i \leqslant n - 1$ from which it follows that

$$\phi_{p_i}(x_i) \leqslant h^{ip_i/n} \quad \text{and} \quad \prod_{i=1}^{n-1} \phi_{p_i}(x_i) \leqslant h^{(1/n)\sum_{i=1}^{n-1} ip_i}.$$

Thus $\sum_{i=1}^{n-1} |x_i|^{n/i} = h > 1$ implies that

$$\sum_{i=1}^{n-1} |x_i|^{n/i} \Big/ \prod_{i=1}^{n-1} \phi_{p_i}(x_i) \geqslant h^{1-1/n\sum_{i=1}^{n-1} ip_i}$$

and the result follows from the assumption that $\sum_{i=1}^{n-1} ip_i < n$.

2. **The Nagumo condition.** In this section we assume that we are dealing with a fixed equation (2) of degree $n \geqslant 3$ and we assume that in (2) the function $f(x, y, y', \ldots, y^{(n-1)})$ is continuous on $I \times R^n$ where I is an interval of the reals.

THEOREM 5. *Assume that the function $f(x, y, y', \ldots, y^{(n-1)})$ in (2) satisfies the condition that given any compact interval $[c, d] \subset I$ and given any $M > 0$ there exist numbers h, p_i, $1 \leqslant i \leqslant n - 1$, such that $h > 0$, $p_i \geqslant 0$ for $1 \leqslant i \leqslant n - 1$, $\sum_{i=1}^{n-1} ip_i < n$, and such that*

$$|f(x, y, y', \ldots, y^{(n-1)})| \leqslant h \prod_{i=1}^{n-1} \phi_{p_i}(y^{(i)}) \tag{3}$$

for all $(x, y, y', \ldots, y^{(n-1)})$ satisfying $c \leqslant x \leqslant d$, $|y| \leqslant M$ where $\phi_p(t)$ is as given in Definition 3. Then each solution of (2) either extends to I or becomes unbounded on its maximal interval of existence.

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PROOF. We prove more than the actual statement of the theorem. We prove that a solution either extends to the right or becomes unbounded on its right maximal interval of existence and that the same result holds for left extensions. Since the proofs are the same we consider only extensions to the right.

Assume that the conditions of the theorem are satisfied but that the conclusion is false. Then there is a compact interval $[c, d] \subset I$ and a solution $y(x)$ of (2) on $[c, d)$ such that $|y(x)|$ is bounded on $[c, d)$ and such that $[c, d)$ is a right maximal interval of existence for $y(x)$. Let $|y(x)| \leqslant M_0$ on $[c, d)$ and let the numbers h and p_i, $1 \leqslant i \leqslant n-1$, be associated with $M_0\,[c, d]$ as in the theorem.

Since $[c, d)$ is right maximal for $y(x)$ it follows that neither $|y^{(n)}(x)|$ nor $|y^{(n-1)}(x)|$ are bounded on $[c, d)$. For $0 \leqslant k \leqslant n$ set

$$M_k(t) = \operatorname{Max}\{|y^{(k)}(x)|: c \leqslant x \leqslant t < d\}.$$

Let t_0 be chosen so that $(c+d)/2 < t_0 < d$ and

$$M_n(t_0) \geqslant 2^n n!\, M_0((d-c)/2)^{-n}.$$

Then for any $t_0 \leqslant t < d$,

$$M_n(t) \geqslant 2^n n!\, M_0((d-c)/2)^{-n} \geqslant 2^n n!\, M_0(t)(t-c)^{-n}$$

and it follows that for any $t_0 \leqslant t < d$

$$M_n'(t) = \operatorname{Max}\{M_n(t), 2^n n!\, M_0(t)(t-c)^{-n}\} = M_n(t).$$

It follows from Lemma 2 that for $t_0 \leqslant t < d$ and for each integer k, $0 < k < n$,

$$\begin{aligned} M_k(t) &\leqslant C_{nk}[M_0(t)]^{1-k/n}[M_n(t)]^{k/n} \\ &\leqslant C_{nk} M_0^{1-k/n}[M_n(t)]^{k/n} = C_{nk}^*[M_n(t)]^{k/n}. \end{aligned}$$

This implies that

$$\sum_{k=1}^{n-1} [M_k(t)]^{n/k} \leqslant \left(\sum_{k=1}^{n-1} (C_{nk}^*)^{n/k}\right) M_n(t) = A_n M_n(t)$$

for $t_0 \leqslant t < d$. However, it follows from inequality (3) that

$$M_n(t) \leqslant h \prod_{k=1}^{n-1} \phi_{p_k}(M_k(t))$$

for all $t_0 \leqslant t < d$. Hence,

$$\sum_{k=1}^{n-1} [M_k(t)]^{n/k} \leqslant hA_n \prod_{k=1}^{n-1} \phi_{p_k}(M_k(t))$$

for $t_0 \leqslant t < d$. This contradicts Lemma 4 since $M_{n-1}(t) \to +\infty$ as $t \to d$ and therefore $\sum_{k=1}^{n-1} [M_k(t)]^{n/k} \to +\infty$ as $t \to d$.

If the nonnegative numbers p_i, $1 \leqslant i \leqslant n-1$, are such that $\sum_{i=1}^{n-1} ip_i > n$, an r with $0 < r < 1$ can be chosen so that $y(x) = -(x_0 - x)^r$ is a solution of

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$$y^{(n)} = h \prod_{i=1}^{n-1} |y^{(i)}|^{p_i} \tag{4}$$

on $(-\infty, x_0)$ provided a suitable value is assigned to the constant h. In fact, substituting $y(x) = -(x_0 - x)^r$ in (4), one can verify that $y(x)$ is a solution on $(-\infty, x_0)$ for a suitable choice of the constant h provided

$$r = \sum_{i=1}^{n-1} ip_i - n \Big/ \sum_{i=1}^{n-1} p_i - 1.$$

Then, using simple induction arguments, one can show that for $n \geqslant 3$ and for nonnegative numbers p_i, $1 \leqslant i \leqslant n-1$, $\sum_{i=1}^{n-1} ip_i \geqslant n$ implies $\sum_{i=1}^{n-1} p_i > 1$ and $\sum_{i=1}^{n-1} ip_i = n$ implies $\sum_{i=2}^{n-1} (i-1)p_i < n-1$. Thus, for each $n \geqslant 3$, nonnegative numbers p_i, $1 \leqslant i \leqslant n-1$, can be chosen so that $0 < r < 1$. It follows that the condition $\sum_{i=1}^{n-1} ip_i < n$ in Theorem 5 could at best possibly be improved to $\sum_{i=1}^{n-1} ip_i \leqslant n$.

Essentially the same arguments as were used in proving Theorem 5 can be used to prove the following more general theorem.

THEOREM 6. *Assume that $f(x, y, y', \ldots, y^{(n-1)})$ satisfies the condition that given any compact interval $[c, d] \subset I$ and given any $M > 0$ there is a function $\phi(t_1, t_2, \ldots, t_{n-1})$ defined for $t_i \geqslant 0$, $1 \leqslant i \leqslant n-1$, such that*

$$|f(x, y, y', \ldots, y^{(n-1)})| \leqslant \phi(|y'|, |y''|, \ldots, |y^{(n-1)}|)$$

for all $(x, y, y', \ldots, y^{(n-1)})$ satisfying $c \leqslant x \leqslant d$, $|y| \leqslant M$, such that $\phi(t_1, t_2, \ldots, t_{n-1})$ is nondecreasing in each variable, and such that

$$\sum_{i=1}^{n-1} t_i^{n/i} \Big/ \phi(t_1, \ldots, t_{n-1}) \to +\infty \quad \text{as} \sum_{i=1}^{n-1} t_i^{n/i} \to +\infty.$$

Then each solution of (2) *either extends to I or becomes unbounded on its maximal interval of existence. In practice Theorem 5 is probably easier to apply.*

Results similar to Theorem 6 for equations of orders 3 and 4 were given in [6].

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