CENTRAL APPROXIMATE UNITS IN A CERTAIN IDEAL OF $L^1(G)$

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Abstract. In this paper we show that for a locally compact group $G$ the ideal $L^0(G) = \{ f | f \in L^1(G), \int f = 0 \}$ of $L^1(G)$ has multiple approximate units belonging to the center of $L^0(G)$ if and only if $G$ has a basis of invariant neighbourhoods of 1 and if all conjugacy classes of $G$ are precompact, or, equivalently, the group of inner automorphisms is precompact in the group of all topological automorphisms. In a sense this is part of the problem to characterize certain classes of groups by properties of the group algebra.

Analogously to the papers of Reiter [7] and Mosak [5] we will characterize the class $[FIA]^{-}$ by the existence of approximate units in a certain ideal of the group algebra. The class $[FIA]^{-}$ consists of all groups such that the group of inner automorphisms is precompact in the group of all continuous automorphisms (see [1, p. 2]). It is known that $[FIA]^{-} = [FC]^{-} \cap [SIN]$, where $[SIN]$ is the class of all groups having a basis of neighbourhoods of 1 invariant under all inner automorphisms, and $[FC]^{-}$ is the class consisting of all groups such that every conjugacy class is precompact (see [2, p. 325, Theorem 4.1]).

In [8] Reiter shows the following theorem:

(1) A locally compact group $G$ has the property $P_1$ (this property is equivalent to the amenability of $G$) iff the ideal $L^0(G)$ consisting of all functions $f$ such that $f \in L^1(G)$ and $\int f = 0$, has multiple, approximate, left units (i.e. there exists a net $(u_\alpha)_{\alpha \in A}$ such that $\lim_{\alpha} u_\alpha * f = f$ for every $f \in L^0(G)$).

In [5] Mosak characterizes the class $[SIN]$ by:

(2) $G \in [SIN]$ iff $L^1(G)$ has bounded, multiple, approximate, two-sided units belonging to the center of $L^1(G)$.

We prove the following

Theorem. The following statements are equivalent:

(i) $G \in [FIA]^{-}$.

(ii) $L^0(G)$ has multiple, approximate units belonging to the center of $L^0(G)$.

Remark. Every abelian or compact and, more generally, every central group is an $[FIA]^{-}$-group.

First we prove

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Proposition. The following statements are equivalent:

(a) \( G \in [FC]^- \).

(b) Given any \( \epsilon > 0 \) and any compact subset \( K \subset G \), there exists a central function \( s \in L^1(G) \) such that \( s \geq 0 \), \( \int s = 1, \| L_y s - s \|_1 < \epsilon, y \in K \), where \( L_y \) denotes the operator on \( L^1(G) \) defined by \( L_y s(x) = s(y^{-1} x) \), for a.e. \( x \in G \).

(c) There exists a net \( (w_\beta)_{\beta \in \Lambda} \) with \( \int w_\beta = 1 \), such that \( w_\beta \) belongs to the center of \( L^1(G) \) and \( \lim_\beta w_\beta \cdot f = 0 \) for all \( f \in L^0(G) \).

Remark. If we omit the property "central" in (b), we obtain just the property \( P_1 \).

Proof. (a) \( \Rightarrow \) (b). First we consider the discrete case. It follows from [3, p. 253, Satz 1], that for every compact, invariant subset \( K_0 \) of a discrete group \( G \in [FC]^- \) the formula

\[
\lim_n m_G(K_0 \cap x^n) / m_G(K_0) = 1
\]

holds, where \( m_G(K_0) \) denotes the Haar measure of the set \( K_0 \). It follows that for every \( \epsilon > 0 \) there exists an integer \( n \in \mathbb{N} \) such that

\[
(+) \quad m_G(K_0 \cap x^n) / m_G(K_0) < \epsilon
\]

(we may assume that \( K_0 \subset K_0^{n+1} \)). Now let \( K \) by any compact subset and \( \epsilon > 0 \); there exists an invariant compact set \( K_0 \) such that \( K_0 \supset K \). Set \( s = m_G(K_0^{n+1})^{-1} \chi_{K_0^n} \), where \( \chi_{K_0^n} \) denotes the characteristic function of \( K_0^n \). Then \( \int s = 1 \) and \( s \) belongs to the center of \( L^1(G) \). By (++) a routine calculation shows that \( \| L_y s - s \|_1 < \epsilon, y \in K \).

In the general case we apply the structure theorem for \([FC]^-\)-groups (see [9, Theorem 3D] and [4]). Hence there exists a compact, normal subgroup \( H \) such that \( G/H = V \times D \), where \( V \) is a vector group and \( D \) is a discrete \([FC]^-\)-group. Now let \( K \) be as before. We denote the canonical map of \( G \) onto \( G/H \) by \( \pi_H \). There exist compact sets \( K_1 \subset V, K_2 \subset D \) such that \( K_1 \times K_2 \supset \pi_H(K) \). The result for discrete groups combined with [7, p. 112, Chapter 5, 2.1(i)], yields the existence of a central function \( \hat{s} \) with \( \hat{s} \in L^1(G/H), \int \hat{s} = 1, \hat{s} \geq 0 \), such that \( \| L_y \hat{s} - \hat{s} \|_1 < \epsilon, y \in K_1 \times K_2 \). If we put \( s = \hat{s} \circ \pi_H \), one can easily see that \( s \) has all desired properties.

(b) \( \Rightarrow \) (c). Denote by \( B \) the set of all pairs \((K, \epsilon)\) with \( K \) compact and \( \epsilon > 0 \). \( B \) is filtering with respect to the relation \( \prec \), where \((K, \epsilon) \prec (K', \epsilon') \) means \( K \subset K' \) and \( \epsilon' < \epsilon \). Now for each \( \beta \in B \) there exists an \( s_{K, \epsilon} \) such that \( \| L_y s_{K, \epsilon} - s_{K, \epsilon} \|_1 < \epsilon, y \in K \). Taking \( w_\beta = s_{K, \epsilon} \), we see by [7, p. 113, Lemma] that \( w_\beta \) belongs to the center of \( L^1(G) \) and that every \( f \in L^0(G) \) satisfies \( \lim_\beta f \ast w_\beta = 0 \).

(c) \( \Rightarrow \) (a). Since the center of \( L^1(G) \) is not trivial (\( w_\beta \neq 0 \)), there exists a compact, invariant neighbourhood \( U \) of the identity (see [5]). Set \( g = m_G(U)^{-1} \chi_U \); it follows that \( v_\beta = w_\beta \ast g \) is a continuous, central function vanishing at infinity. Furthermore \( \int v_\beta = 1 \) and

\[
(++) \quad \lim_\beta \| f \ast v_\beta \|_1 = \lim_\beta \| f \ast w_\beta \ast g \|_1 \leq \lim_\beta \| f \ast w_\beta \|_1 = 0.
\]

Let \( H \) be the normal subgroup consisting of all elements with precompact
conjugacy class. \( H \) is an open subgroup of \( G \) since \( U \subset H \). It is known that 
\( f \in L^1(G) \) is central iff \( f \) satisfies, for all \( x \in G \), \( f(xy^{-1}) = f(y)\Delta(x) \) a.e. \( y \in G \), where the null set may depend on \( x \) (see [6, Proposition 1.2]). Since the center of \( L^1(G) \) is not trivial, \( G \) is unimodular [5]. Hence we obtain that 
\( v_{\beta}(xy^{-1}) = v_{\beta}(y) \) for all \( x, y \in G \), because \( v_{\beta} \) is continuous, \( \beta \in B \). Therefore the conjugacy class of each \( y \in G \) satisfying \( v_{\beta}(y) \neq 0 \) for any \( \beta \in B \) is precompact (\( v_{\beta} \) is continuous and vanishes at infinity); in particular \( \text{Supp} v_{\beta} \subset H \). If \( G \neq H \), we take an element \( a \in G \) such that \( aH \cap H = \emptyset \).

Choosing any \( f_1 \in L^1(G) \) satisfying \( f_1 \geq 0, \int f_1 = 1, \text{Supp} f_1 \subset H \), we put \( f = L_\alpha f - f \) and obtain

\[
\text{Supp}(v_{\beta} \ast L_\alpha f_1) \cap \text{Supp}(v_{\beta} \ast f_1) \subset H \cdot aH \cap H \cdot H = \emptyset.
\]

Hence we get

\[
\|f \ast v_{\beta}\|_1 = \|v_{\beta} \ast L_\alpha f_1\|_1 + \|v_{\beta} \ast f_1\|_1 \geq (\int v_{\beta}) \cdot (\int f_1) = 1
\]

in contradiction to \((+++\)) Therefore \( G = H \), i.e. \( G \in [FC]^\sim \).

Now we are able to prove the theorem.

(i) \( \Rightarrow \) (ii). (2) yields the existence of bounded, multiple, approximate, two-sided units \((u_a)_{a \in A}\) belonging to the center of \( L^1(G) \) such that \( \int u_a = 1, a \in A \). Since \( G \in [FC]^\sim \), it follows by the proposition above that there exists a net \((w_{\beta})_{\beta \in B}\) in the center of \( L^1(G) \) with \( \int w_{\beta} = 1 \) and \( \lim_{\beta} w_{\beta} \ast f = 0, f \in L^0(G) \). If we put \( u_{a,\beta} = u_a - w_{\beta} \), one can easily see that \( u_{a,\beta} \) belongs to the center of \( L^0(G) \), and, moreover, every \( f \in L^0(G) \) satisfies

\[
\lim_{a,\beta} f \ast u_{a,\beta} = \lim_{\alpha} f \ast u_{\alpha} - \lim_{\beta} f \ast w_{\beta} = f,
\]

i.e. \((u_{a,\beta})_{a \in A, \beta \in B}\) is the desired approximate unit.

(ii) \( \Rightarrow \) (i). First we show that \( G \in [SIN] \). Let \( f \) belong to the center of \( L^0(G) \). We want to show that \( f \ast g = g \ast f \) holds for every \( g \in L^1(G) \), i.e. \( f \) is in the center of \( L^1(G) \). We assume that \( \int g = 1 \). Let \((v_{\beta})_{\beta \in B}\) be a bounded, multiple, approximate, two-sided unit in \( L^1(G) \) with \( \int v_{\beta} = 1, \beta \in B \). Hence we obtain

\[
f \ast g - g \ast f = \lim_{\beta} f \ast (g - v_{\beta}) - \lim_{\beta}(g - v_{\beta}) \ast f
\]

\[
+ \lim_{\beta} f \ast v_{\beta} - \lim_{\beta} v_{\beta} \ast f = 0,
\]

because \( g - v_{\beta} \) belongs to \( L^0(G), \beta \in B \). This fact implies that \((u_q)\) is contained in the center of \( L^1(G) \). Next we observe that the operator on \( L^1(G) \) of left translation by \( x \) cannot reduce to the identity on \((u_q)\) (otherwise \( L_xf = f \) holds for any \( f \in L^0(G) \); a trivial modification of the proof in [5, p. 615] shows that this is a contradiction). Now one can imitate step by step the proof of [5]. Therefore \( G \in [SIN] \). To show that \( G \in [FC]^\sim \) it suffices, by the proposition, to show that there exists a net \((w_{\gamma})_{\gamma \in \mathcal{C}}\) belonging to the center of \( L^1(G) \) such that \( \int w_{\gamma} = 1 \) and \( \lim_{\gamma} w_{\gamma} \ast f = 0, f \in L^0(G) \). If we put \( w_{a,\beta} = v_{\beta} - u_a \) then \( w_{a,\beta} \) belongs to the center of \( L^1(G) \); furthermore \( \int w_{a,\beta} = 1 \).
and \( \lim_{\alpha,\beta} w_{\alpha,\beta} \ast f = 0, f \in L^0(G) \). Hence \( G \in [FC]^- \cap [SIN] \), i.e. \( G \in [FIA]^- \).

The methods of our proof admit the following

**Corollary.** If \( L^0(G) \) possesses central approximate units, then \( L^0(G) \) possesses even bounded, multiple, approximate units consisting of continuous functions with compact support, the norm of the units being bounded by the constant 2.

**Bibliography**


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